Solutions

MATH 155
Chapter 10 Practice Test

Name

Use Algebraic Notation AND Show All of Your Work

1. **(a)** Find the ***STANDARD FORM*** equation of the conic that has its vertex at $(-1,2)$ and its focus at $(-1,0)$. **(b)** Graph this conic. **(c)** Find the equation of the directrix. **(d) Label** the vertex, focus, and directrix on your graph. *(Be careful with your notation, and show your steps clearly.)*

(a)

$(x-h)^2 = 4p(y-k)$

(h,k) = vertex

p = directed distance from vertex to focus

Vertex:

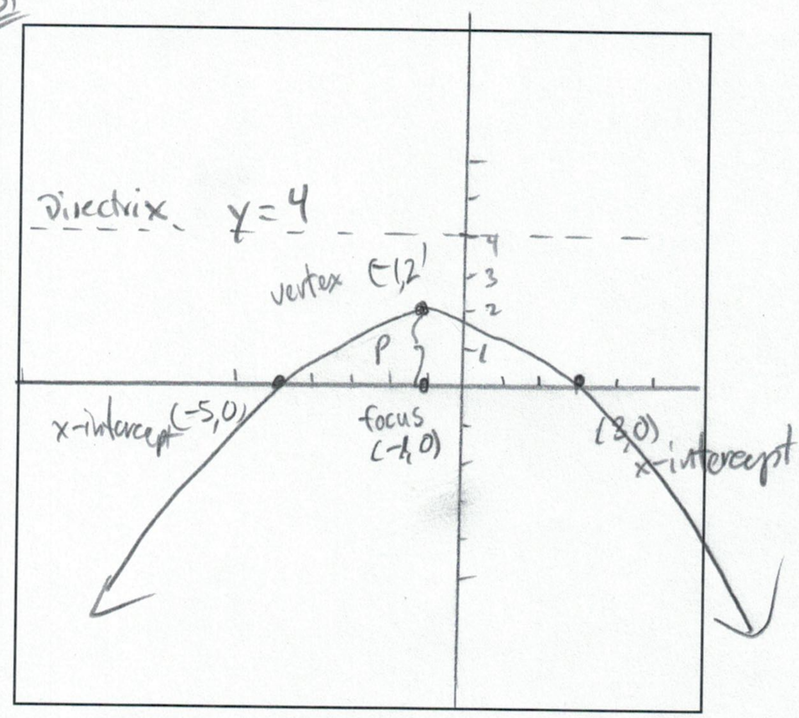
$(h,k) = (-1,2)$

$p = -2$, opens down

$[x-(-1)]^2 = 4(-2)[y-(2)]$

$(x+1)^2 = -8(y-2)$

(b)



(a) Standard Form of conic: $(x+1)^2 = -8(y-2)$

(c) equation of directrix : $y = 4$

2. **(a)** Find the ***STANDARD FORM*** equation of the conic that the following equation: $y^2+6y+8x+25=0$. **(b)** Graph this conic. **(c)** Find the vertex, focus, and directrix. **(d) Label** the vertex, focus, and directrix on your graph. *(Be careful with your notation, and show your steps clearly.)*

(a)

$y^2+6y+8x+25=0$

$y^2+6y=-8x-25$

$y^2+6y+9=-8x-25+9$

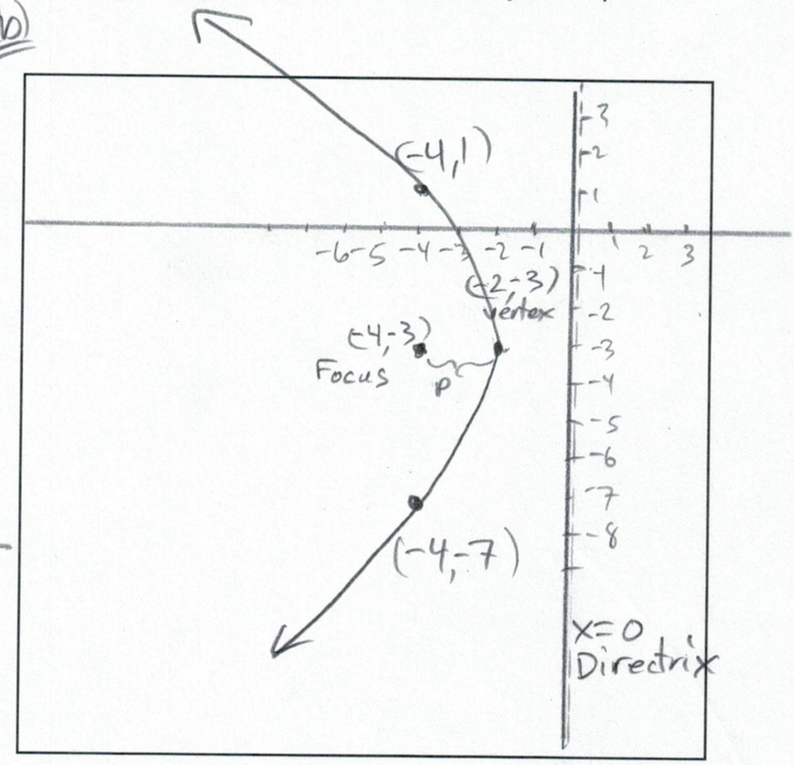
$\left[\frac{1}{2}(6)\right]^2=(3)^2=9$

$(y+3)^2=-8x-16$

$\boxed{(y+3)^2=-8(x+2)}$

$(y-k)^2=4p(x-h)$

(b)



(c)

Vertex:

$(h,k)=(-2,-3)$

$4p=-8$

$p=-2$, opens left

Focus:

$(h+p,k)=(-2+(-2),-3)$

$=(-4,-3)$

(a) Standard Form of conic: $(y+3)^2=-8(x+2)$

(c) vertex: $(-2,-3)$ focus: $(-4,-3)$

(c) equation of directrix: $x=0$

3. **(a)** Rewrite the equation of the conic $9x^2+4y^2+36x-24y+36=0$ in ***STANDARD FORM***. **(b)** Sketch a graph of this conic. **(c)** Find the center, foci, and vertices. **(d) Label** the center, foci, and vertices
on your graph. *(Be careful with your notation, and show your steps clearly.)*

(a)

$9x^2+4y^2+36x-24y+36=0$

$(9x^2+36x)+(4y^2-24y)=-36$

$9(x^2+4x)+4(y^2-6y)=-36$

$9(x^2+4x+4)+4(y^2-6y+9)=-36+36+36$

$[\frac{1}{2}(4)]^2=(2)^2=4$, $[\frac{1}{2}(-6)]^2=(-3)^2=9$

$9(x+2)^2+4(y-3)^2=36$

$\frac{9(x+2)^2}{36}+\frac{4(y-3)^2}{36}=\frac{36}{36}$

$\boxed{\frac{(x+2)^2}{4}+\frac{(y-3)^2}{9}=1}$

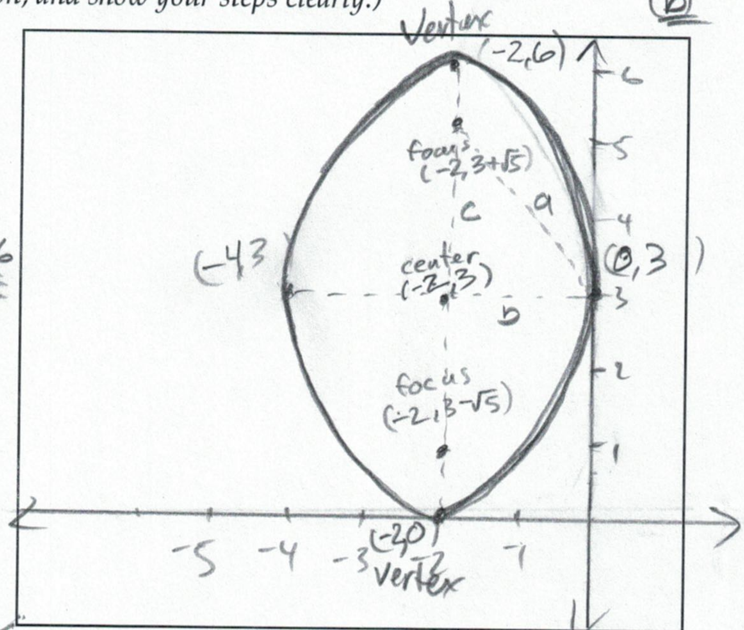
$a^2>b^2$

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

$a^2=9,\ b^2=4$
$a=3,\ b=2$

$a^2=b^2+c^2$
$9=4+c^2$
$5=c^2$
$c=\sqrt{5}$

(b)



(c)

Center:
$(h,k)=(-2,3)$

Vertices:
$(h,k+a)=(-2,3+3)=(-2,6)$
$(h,k-a)=(-2,3-3)=(-2,0)$

Foci:
$(h,k+c)=(-2,3+\sqrt{5})$
$(h,k-c)=(-2,3-\sqrt{5})$

(a) Standard Form of conic: $\frac{(x+2)^2}{4}+\frac{(y-3)^2}{9}=1$

(c) center: $(-2,3)$ foci: $(-2,3+\sqrt{5})$ & $(-2,3-\sqrt{5})$

vertices: $(-2,6)$ & $(-2,0)$

4. **(a)** Rewrite the equation of the conic $3x^2-2y^2-6x-12y-27=0$ in ***STANDARD FORM***. **(b)** Sketch a graph of this conic. **(c)** Find the center, foci, vertices, and the equations of the asymptotes. **(d) Label** the center, foci, vertices, and the asymptotes on your graph. *(Be careful with your notation, and show your steps clearly.)*

(a)

$3x^2-2y^2-6x-12y-27=0$

$(3x^2-6x)+(-2y^2-12y)=27$

$3(x^2-2x)-2(y^2+6y)=27$

$3(x^2-2x+\underline{\underline{1}})-2(y^2+6y+\underline{\underline{9}})=27+\underline{\underline{3}}-\underline{\underline{18}}$

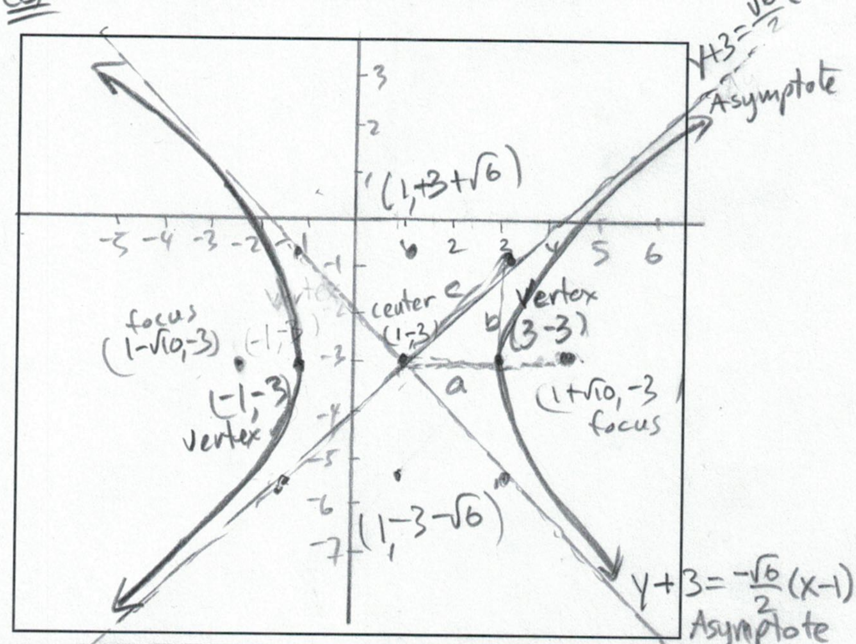
$\left[\tfrac{1}{2}(-2)\right]^2=(-1)^2=\underline{\underline{1}},\ \left[\tfrac{1}{2}(6)\right]^2=(3)^2=\underline{\underline{9}}$

$3(x-1)^2-2(y+3)^2=12$

$\dfrac{3(x-1)^2}{12}-\dfrac{2(y+3)^2}{12}=\dfrac{12}{12}$

$$\boxed{\frac{(x-1)^2}{4}-\frac{(y+3)^2}{6}=1}$$

(b)



(c)

$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$

$a^2=4,\ b^2=6$

$a=2,\ b=\sqrt{6}$

$a^2+b^2=c^2$

$4+6=c^2$

$10=c^2$

$\sqrt{10}=c$

Vertices:

$(h+a,k)=(1+2,-3)=(3,-3)$

$(h-a,k)=(1-2,-3)=(-1,-3)$

Foci:

$(h+c,k)=(1+\sqrt{10},-3)$

$(h-c,k)=(1-\sqrt{10},-3)$

Asymptotes:

$y-y_1=m(x-x_1)$

$(x_1,y_1)=(h,k)=(1,-3)$

$m=\pm\dfrac{b}{a}=\pm\dfrac{\sqrt{6}}{2}$

$y-(-3)=\pm\dfrac{\sqrt{6}}{2}[x-(1)]$

$y+3=\pm\dfrac{\sqrt{6}}{2}(x-1)$

Center:

$(h,k)=(1,-3)$

(a) Standard Form of conic: $\dfrac{(x-1)^2}{4}-\dfrac{(y+3)^2}{6}=1$

(c) center : $(1,-3)$ foci : $(1+\sqrt{10},-3)$ & $(1-\sqrt{10},-3)$

vertices : $(3,-3)$ & $(-1,-3)$ asymptotes: $y+3=\pm\dfrac{\sqrt{6}}{2}(x-1)$

5. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = t^3 \\ y = \frac{1}{2}t^2 \end{cases}$. **(b)** Eliminate the parameter and write the corresponding rectangular equation. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

Solve for t:

$x = t^3$

$\sqrt[3]{x} = t$

$y = \frac{1}{2}t^2$

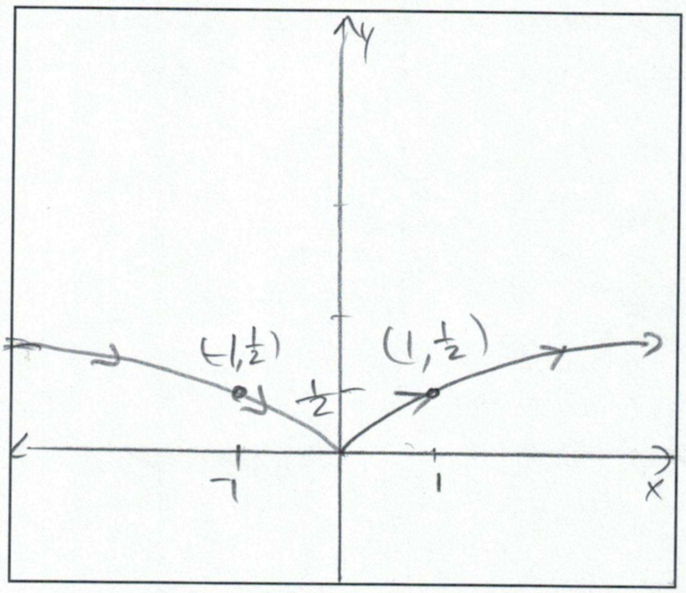
substitute for t.

$y = \frac{1}{2}\left[\sqrt[3]{x}\right]^2$

$y = \frac{1}{2}\left[x^{1/3}\right]^2$

$y = \frac{1}{2}x^{2/3}$

(a)



6. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = 4 + 2\cos(\theta) \\ y = -1 + 4\sin(\theta) \end{cases}$. **(b)** Eliminate the parameter and write the corresponding rectangular equation in *STANDARD FORM*. **(c)** State the center and vertices. **(d) Label** the center and vertices on your graph. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

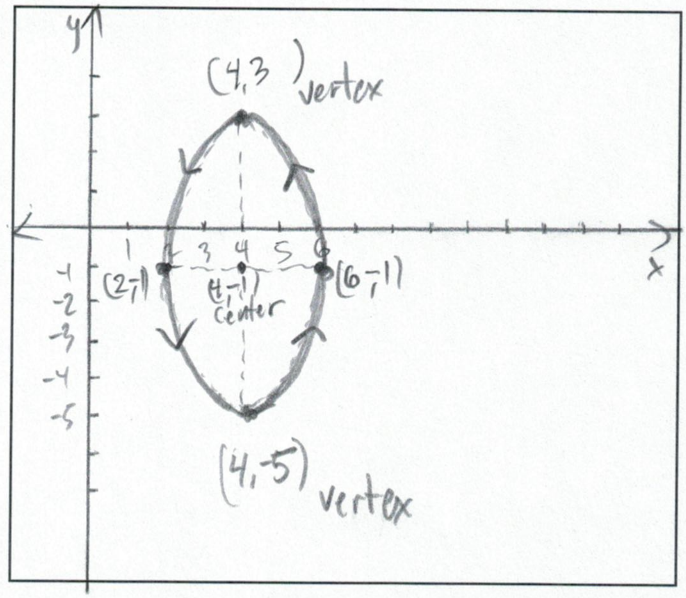
$x = 4 + 2\cos(\theta)$	$y = -1 + 4\sin(\theta)$
$x - 4 = 2\cos(\theta)$	$y + 1 = 4\sin(\theta)$
$\frac{(x-4)}{2} = \cos(\theta)$	$\frac{y+1}{4} = \sin(\theta)$
$\left(\frac{x-4}{2}\right)^2 = \cos^2(\theta)$	$\left(\frac{y+1}{4}\right)^2 = \sin^2(\theta)$
$\frac{(x-4)^2}{4} = \cos^2(\theta)$	$\frac{(y+1)^2}{16} = \sin^2(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$$\boxed{\frac{(y+1)^2}{16} + \frac{(x-4)^2}{4} = 1}$$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \quad a^2 > b^2$$

(a)



(c)

Center:
$(h, k) = (4, -1)$

Vertices:
$(h, k+a) = (4, -1+4) = (4, 3)$
$(h, k-a) = (4, -1-4) = (4, -5)$

(b) Standard Form of conic: $\frac{(y+1)^2}{16} + \frac{(x-4)^2}{4} = 1$

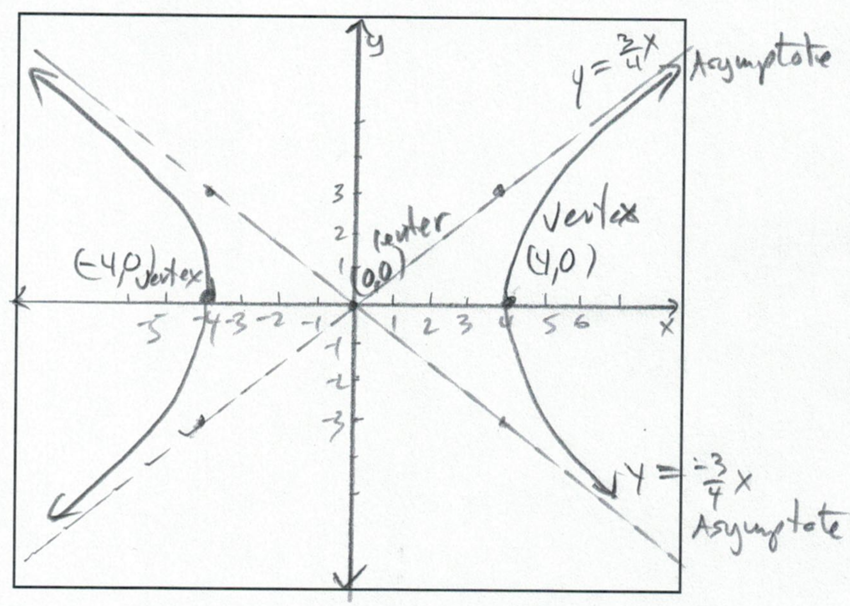
(c) center : $(4, -1)$ vertices : $(4, 3)$ & $(4-5)$

7. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = 4\sec(\theta) \\ y = 3\tan(\theta) \end{cases}$. **(b)** Eliminate the parameter and write the corresponding rectangular equation in *STANDARD FORM*. **(c)** State the center and vertices. **(d) Label** the center and vertices on your graph. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

$x = 4\sec(\theta)$	$y = 3\tan(\theta)$
$\frac{x}{4} = \sec(\theta)$	$\frac{y}{3} = \tan(\theta)$
$\left(\frac{x}{4}\right)^2 = \sec^2(\theta)$	$\left(\frac{y}{3}\right)^2 = \tan^2(\theta)$
$\frac{x^2}{16} = \sec^2(\theta)$	$\frac{y^2}{9} = \tan^2\theta$

(a)



$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \frac{y^2}{9} = \frac{x^2}{16}$$

$$\boxed{1 = \frac{x^2}{16} - \frac{y^2}{9}}$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

c) $a^2 = 16$, $b^2 = 9$

$a = 4$, $b = 3$

Center:
$(h,k) = (0,0)$

Vertices:
$(h+a, k) = (0+4, 0) = (4,0)$
$(h-a, k) = (0-4, 0) = (-4,0)$

(b) Standard Form of conic: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(c) center : $(0,0)$ vertices : $(4,0)$ & $(-4,0)$

8. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = t+1 \\ y = t^2+3t \end{cases}$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find $\frac{d^2y}{dx^2}$. **(d)** Use $\frac{dy}{dx}$ to find slope of the tangent line when $t=-1$. **(e)** Use $\frac{d^2y}{dx^2}$ to find the concavity of the curve when $t=-1$. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(b)

$\frac{dx}{dt} = \frac{d}{dt}[t+1]$

$\frac{dx}{dt} = 1$

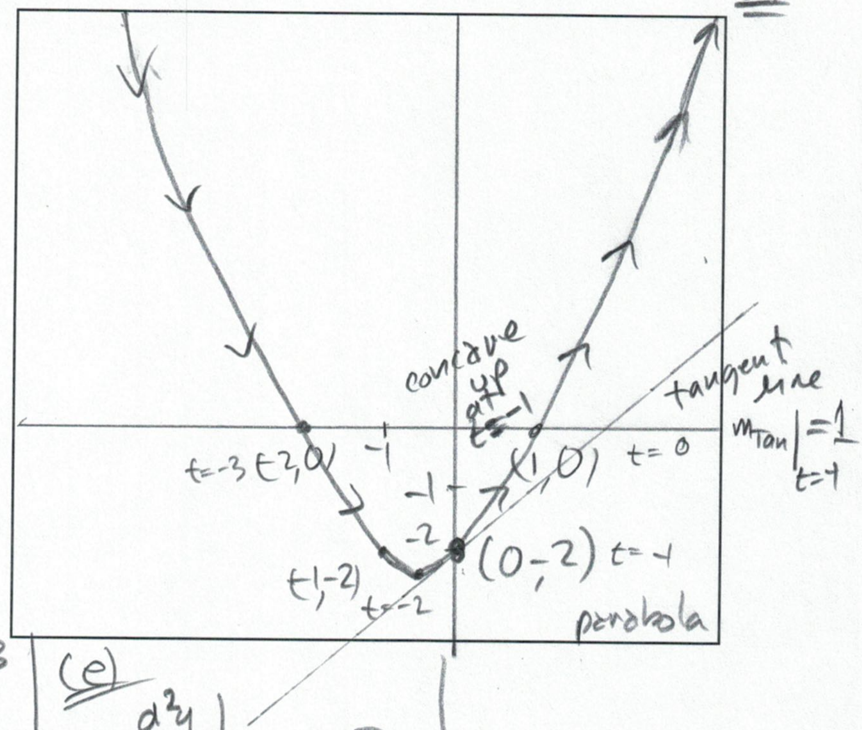
$\frac{dy}{dt} = \frac{d}{dt}[t^2+3t]$

$\frac{dy}{dt} = 2t+3$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{dy}{dx} = \frac{2t+3}{1}$

$\boxed{\frac{dy}{dt} = 2t+3}$

(a)



(c)

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[2t+3]}{1}$

$\frac{d^2y}{dx^2} = \frac{2}{1}$

$\frac{d^2y}{dx^2} = 2$

(d)

$\frac{dy}{dx}\Big|_{t=-1} = 2(-1)+3$

$m_{Tan}\Big|_{t=-1} = -2+3$

$m_{Tan}\Big|_{t=-1} = 1$

(e)

$\frac{d^2y}{dx^2}\Big|_{t=-1} = 2$

$\frac{d^2y}{dx^2}\Big|_{t=-1} = 2 > 0$

"Concave up"

(b) $\frac{dy}{dx} =$ $2t+3$

(c) $\frac{d^2y}{dx^2} =$ 2

(d) slope of the tangent line when $t=-1$: $\frac{dy}{dx}\Big|_{t=-1} = 1 = m_{Tan}\Big|_{t=-1}$

(e) concavity of the curve when $t=-1$: concave up $\frac{d^2y}{dx^2}\Big|_{t=-1} = 2$

9. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) \end{cases}$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find $\frac{d^2y}{dx^2}$. **(d)** Use $\frac{dy}{dx}$ to find slope of the tangent line when $t = \frac{\pi}{4}$. **(e)** Use $\frac{d^2y}{dx^2}$ to find the concavity of the curve when $t = \frac{\pi}{4}$. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(b)

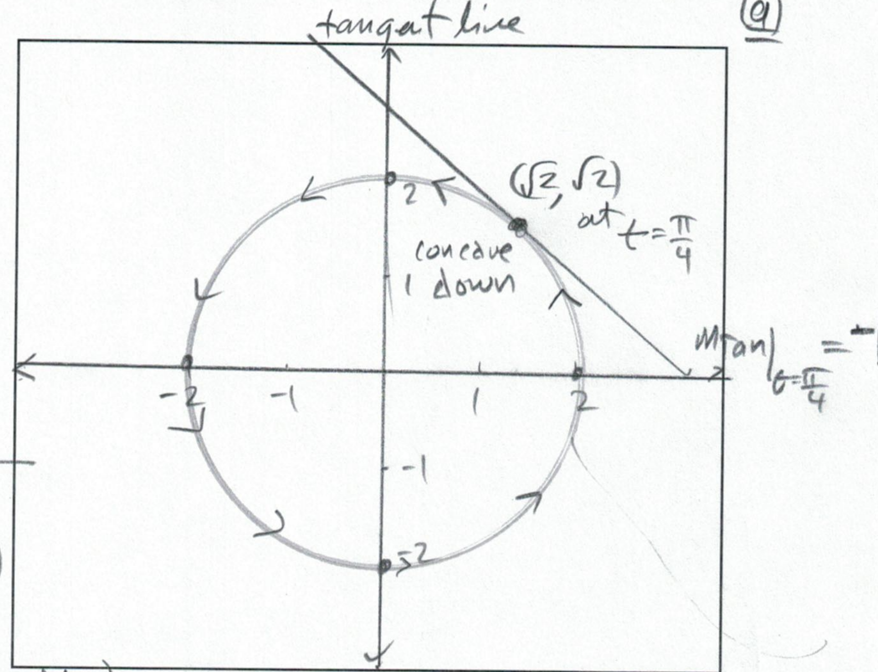
$\frac{dx}{d\theta} = \frac{d}{d\theta}[2\cos(\theta)]$ | $\frac{dy}{d\theta} = \frac{d}{d\theta}[2\sin(\theta)]$

$\frac{dx}{d\theta} = -2\sin(\theta)$ | $\frac{dy}{d\theta} = 2\cos(\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, $\frac{dy}{dx} = \frac{2\cos(\theta)}{-2\sin(\theta)}$

$\frac{dy}{dx} = -\cot(\theta)$

(a)



(c)

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-\cot(\theta)]}{-2\sin(\theta)}$

$\frac{d^2y}{dx^2} = \frac{-(-\csc^2(\theta))}{-2\sin(\theta)}$

$\frac{d^2y}{dx^2} = -\frac{1}{2}\csc^3(\theta)$

(d)

$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right)$

$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = -1$

$m_{Tan}\Big|_{\theta=\frac{\pi}{4}} = -1$

(e)

$\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = -\frac{1}{2}\csc^3\left(\frac{\pi}{4}\right)$

$\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = -\frac{1}{2}\left[\sqrt{2}\right]^3$

$\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = \frac{-2\sqrt{2}}{2}$

$\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = -\sqrt{2} < 0$

"Concave down"

(b) $\frac{dy}{dx} = -\cot(\theta)$

(c) $\frac{d^2y}{dx^2} = -\frac{1}{2}\csc^3(\theta)$

(d) slope of the tangent line when $t = \frac{\pi}{4}$: $\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = -1 = m_{Tan}\Big|_{t=\frac{\pi}{4}}$

(e) concavity of the curve when $t = \frac{\pi}{4}$: concave down $\frac{d^2y}{dx^2}\Big|_{\theta=\frac{\pi}{4}} = -\sqrt{2}$

10. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = t^2 - t - 2 \\ y = t^3 - 3t \end{cases}$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find all point(s) of ***horizontal*** tangency. **(d)** Find all point(s) of ***vertical*** tangency. **(e) Label** the point(s) of ***horizontal*** tangency and the point(s) ***vertical*** tangency on the curve. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(b)

$\frac{dx}{dt} = \frac{d}{dt}(t^2 - t - 2)$

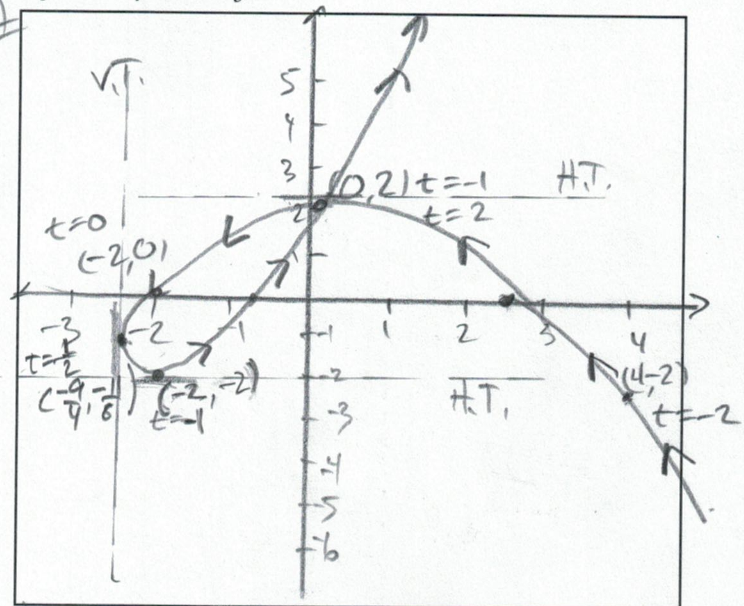
$\frac{dx}{dt} = 2t - 1$

$\frac{dy}{dt} = \frac{d}{dt}(t^3 - 3t)$

$\frac{dy}{dt} = 3t^2 - 3$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$

(a)



(c)

Horizontal Tangency:

$\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 0$

$3t^2 - 3 = 0$

$3(t^2 - 1) = 0$

$3(t+1)(t-1) = 0$

Either

$t + 1 = 0$, or $t - 1 = 0$

$t = -1$, $t = 1$

(d)

Vertical Tangency:

$\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

$\frac{dx}{dt} = 0$

$2t - 1 = 0$

$2t = 1$

$t = \frac{1}{2}$

(c)

$t = -1$	$t = 1$
$x(-1) = 0$ $y(-1) = 2$	$x(1) = -2$ $y(1) = -2$
at $(0,2)$ Horizontal Tangent since $\left.\frac{dy}{dt}\right\rvert_{t=-1} = 0$ and $\left.\frac{dx}{dt}\right\rvert_{t=-1} \neq 0$	at $(-2,-2)$ Horizontal Tangent since $\left.\frac{dy}{dt}\right\rvert_{t=1} = 0$ and $\left.\frac{dx}{dt}\right\rvert_{t=1} \neq 0$

(d) $t = \frac{1}{2}$

$x(\frac{1}{2}) = -2.25$

$y(\frac{1}{2}) = -1.375$

at $(-2.25, -1.375)$ Vertical Tangent since $\left.\frac{dx}{dt}\right\rvert_{t=\frac{1}{2}} = 0$ and $\left.\frac{dy}{dt}\right\rvert_{t=\frac{1}{2}} \neq 0$

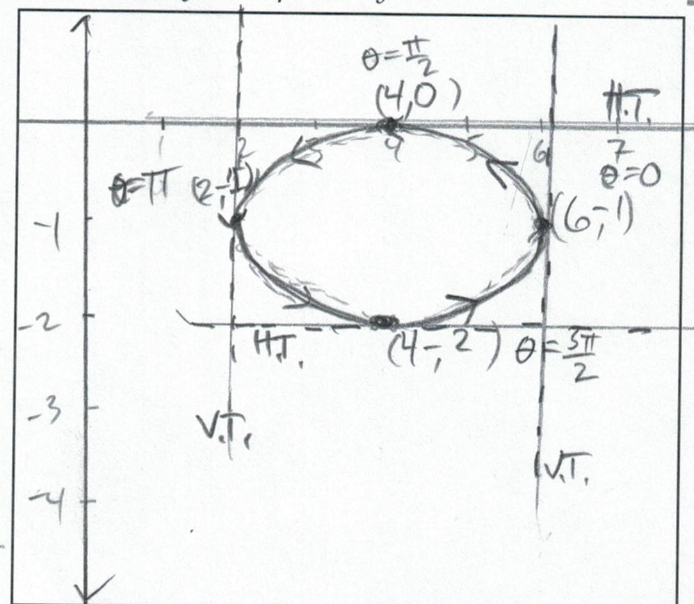
(b) $\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$ or $\frac{3(t^2 - 1)}{2t - 1}$

(c) point(*s*) of ***horizontal*** tangency: $(0,2)$ & $(-2,-2)$

(d) point(*s*) of ***vertical*** tangency: $(-2.25, -1.375)$ or $\left(\frac{-9}{4}, \frac{-11}{8}\right)$

11. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = 4 + 2\cos(\theta) \\ y = -1 + \sin(\theta) \end{cases}$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find all point(*s*) of ***horizontal*** tangency. **(d)** Find all point(*s*) of ***vertical*** tangency. **(e) Label** the point(*s*) of ***horizontal*** tangency and the point(*s*) ***vertical*** tangency on the curve. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(a)



(b)

$\frac{dx}{d\theta} = \frac{d}{d\theta}[4+2\cos(\theta)]$ | $\frac{dy}{d\theta} = \frac{d}{d\theta}[-1+\sin(\theta)]$

$\frac{dx}{d\theta} = -2\sin(\theta)$ | $\frac{dy}{d\theta} = \cos(\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, $\frac{dy}{dx} = \frac{\cos(\theta)}{-2\sin(\theta)}$

$\frac{dy}{dx} = -\frac{1}{2}\cot(\theta)$

(c) Horizontal Tangency:

$\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$

$\frac{dy}{d\theta} = 0$

$0 = \cos(\theta)$

$\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

(d) Vertical Tangency:

$\frac{dx}{d\theta} = 0$ & $\frac{dy}{d\theta} \neq 0$

$\frac{dx}{d\theta} = 0$

$0 = -2\sin(\theta)$

$0 = \sin(\theta)$

$\theta = 0$, or $\theta = \pi$

If $\theta=0$,	If $\theta=\pi$,	If $\theta=\frac{\pi}{2}$,	If $\theta=\frac{3\pi}{2}$,
$x(0)=6$ $y(0)=-1$	$x(\pi)=2$ $y(\pi)=-1$	$x(\frac{\pi}{2})=4$ $y(\frac{\pi}{2})=0$	$x(\frac{3\pi}{2})=4$ $y(\frac{3\pi}{2})=-2$
at (6,-1) Vertical Tangent since $\left.\frac{dx}{d\theta}\right\|_{\theta=0}=0$ and $\left.\frac{dy}{d\theta}\right\|_{\theta=0}\neq 0$	at (2,-1) Vertical Tangent since $\left.\frac{dx}{d\theta}\right\|_{\theta=\pi}=0$ and $\left.\frac{dy}{d\theta}\right\|_{\theta=\pi}\neq 0$	at (4,0) Horizontal Tangent since $\left.\frac{dy}{d\theta}\right\|_{\theta=\frac{\pi}{2}}=0$ and $\left.\frac{dx}{d\theta}\right\|_{\theta=\frac{\pi}{2}}\neq 0$	at (4,-2) Horizontal Tangent since $\left.\frac{dy}{d\theta}\right\|_{\theta=\frac{3\pi}{2}}=0$ and $\left.\frac{dx}{d\theta}\right\|_{\theta=\frac{3\pi}{2}}\neq 0$

(b) $\frac{dy}{dx} = -\frac{1}{2}\cot(\theta)$

(c) point(*s*) of ***horizontal*** tangency: (4,0) & (4,-2)

(d) point(*s*) of ***vertical*** tangency: (6,-1) & (2,-1)

12. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = 2t - t^2 \\ y = 2t^{\frac{2}{3}} \end{cases}$ over the interval $1 \le t \le 2$. **(b)** Write an integral that represents the arc length of this curve over the interval $1 \le t \le 2$. ***(Do not attempt to evaluate this integral algebraically.)*** **(c)** Use the **numerical integration capability** of a graphing utility to approximate the value of this integral. Round your result to the nearest **tenth.** *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

$$\text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[2t - t^2\right]$$

$$\frac{dx}{dt} = 2 - 2t$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[2t^{2/3}\right]$$

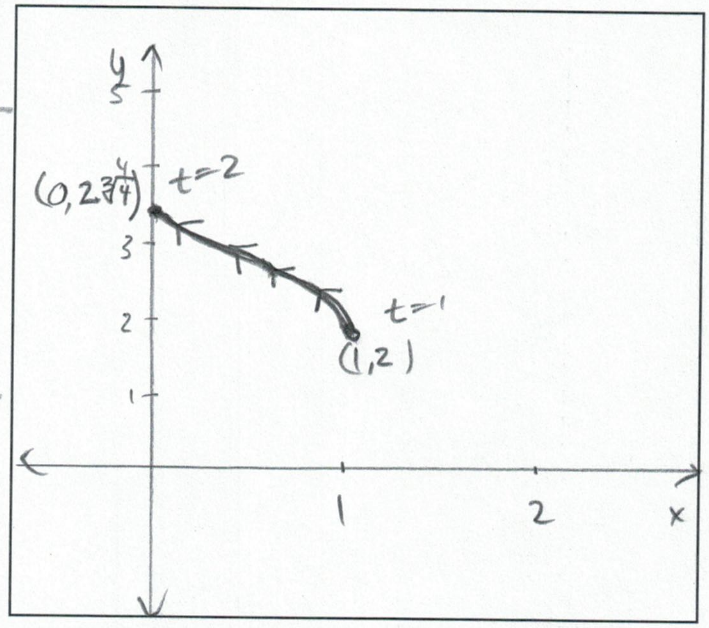
$$\frac{dy}{dt} = 2 \cdot \frac{2}{3} t^{-1/3}$$

$$\frac{dy}{dt} = \frac{4}{3} t^{-1/3}$$

$$\text{Arc Length} = \int_1^2 \sqrt{(2-2t)^2 + \left(\frac{4}{3}t^{-1/3}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{4 - 8t + 4t^2 + \frac{16}{9}t^{-2/3}}\, dt$$

(a)



(c)

≈ 1.6219

≈ 1.6

(b) arc length integral = $\int_1^2 \sqrt{4 - 8t + 4t^2 + \frac{16}{9}t^{-2/3}}\, dt$

(c) arc length *approximation* ≈ 1.6

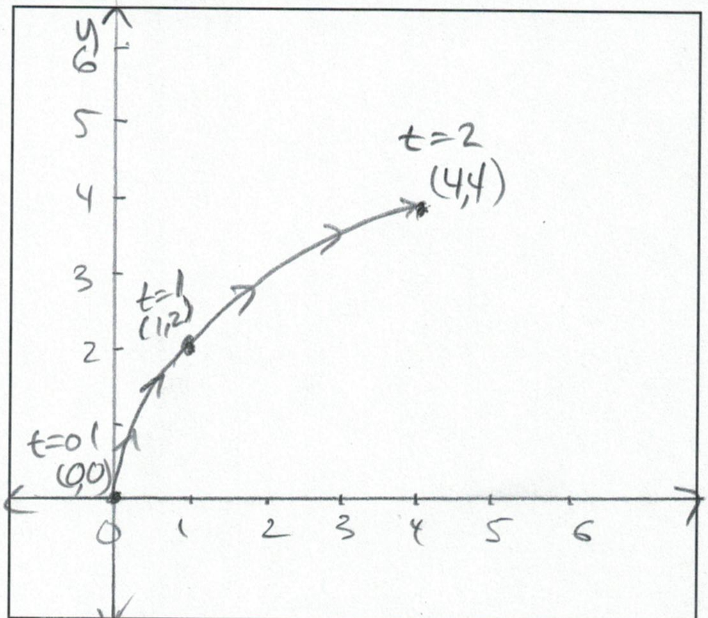
13. **(a)** Use a graphing utility to graph the curve represented by the following parametric equations: $\begin{cases} x = t^2 \\ y = 2t \end{cases}$ over the interval $0 \le t \le 2$. **(b)** Write an integral that represents the arc length of this curve over the interval $0 \le t \le 2$. **(c)** Use a table of integrals to complete the computation of this arc length integral, and the **numerical integration capability** of a graphing utility to approximate the value of this integral. Round your result to the nearest **tenth**. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

(a)

Arc Length $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dx}{dt} = \frac{d}{dt}[t^2]$	$\frac{dy}{dt} = \frac{d}{dt}[2t]$
$\frac{dx}{dt} = 2t$	$\frac{dy}{dt} = 2$



Arc Length $= \int_0^2 \sqrt{(2t)^2 + (2)^2}\, dt$

$= \int_0^2 \sqrt{4t^2 + 4}\, dt$

$= \int_0^2 \sqrt{4}\sqrt{t^2+1}\, dt$

$= 2\int_0^2 \sqrt{t^2+1}\, dt$

See page A4 Integration Tables #26

(c)

$= 2 \cdot \frac{1}{2}\left[t\sqrt{t^2+1} + \ln\left|t + \sqrt{t^2+1}\right|\right]_0^2$

$= \left((2)\sqrt{(2)^2+1} + \ln\left|(2) + \sqrt{(2)^2+1}\right|\right) - \left((0)\sqrt{(0)^2+1} + \ln\left|(0) + \sqrt{(0)^2+1}\right|\right)$

$= \left(2\sqrt{5} + \ln\left|2+\sqrt{5}\right|\right) - \left(0 + \ln(1)\right)$

$= 2\sqrt{5} + \ln(2+\sqrt{5}) - 0$

$= 2\sqrt{5} + \ln(2+\sqrt{5}) \approx 5.91577 \approx 5.9$

(b) arc length integral = $\int_0^2 \sqrt{4t^2+4}\, dt$ or $2\int_0^2 \sqrt{t^2+1}\, dt$

(c) arc length *approximation* ≈ 5.9

$x = r\cos(\theta)$, $y = r\sin(\theta)$

14. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta) = 2\cos(\theta)$ over the interval $0 \le \theta < \pi$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find all points of ***horizontal*** tangency. **(d)** Find all points of ***vertical*** tangency. **(e) Label** these points of ***horizontal*** tangency and the points of ***vertical*** tangency on the curve. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(b)

$\frac{dr}{d\theta} = \frac{d}{d\theta}[2\cos(\theta)] = 2(-\sin(\theta)) = -2\sin(\theta)$

$\frac{dy}{d\theta} = \frac{d}{d\theta}[r\sin(\theta)] = r\cos(\theta) + \sin\theta \cdot \frac{dr}{d\theta}$

$\frac{dy}{d\theta} = [2\cos(\theta)]\cdot[\cos(\theta)] + [\sin(\theta)]\cdot[-2\sin(\theta)]$

$\frac{dy}{d\theta} = 2\cos^2(\theta) - 2\sin^2(\theta)$

$\frac{dy}{d\theta} = 2[\cos^2(\theta) - \sin^2(\theta)]$

$\frac{dy}{d\theta} = 2\cos(2\theta)$

$\frac{dx}{d\theta} = \frac{d}{d\theta}[r\cos(\theta)] = r(-\sin(\theta)) + \cos(\theta)\cdot\frac{dr}{d\theta}$

$\frac{dx}{d\theta} = [2\cos(\theta)]\cdot[-\sin(\theta)] + [\cos(\theta)]\cdot[-2\sin(\theta)]$

$\frac{dx}{d\theta} = -2\sin(\theta)\cos(\theta) - 2\sin(\theta)\cos(\theta)$

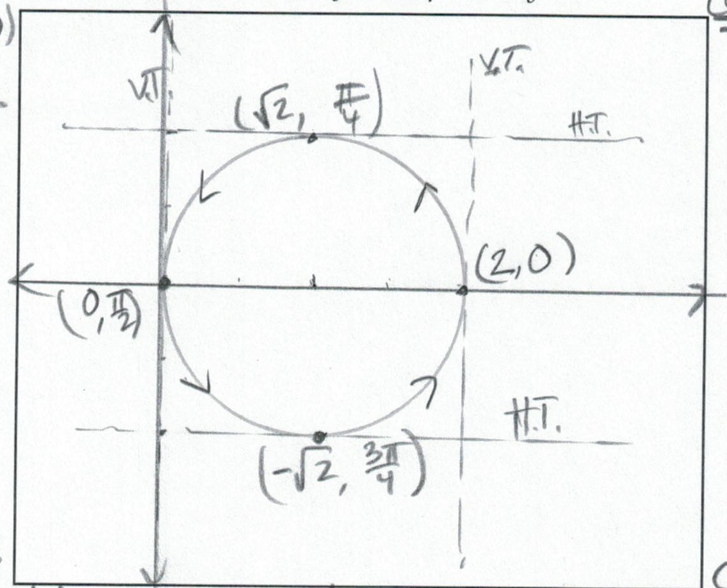
$\frac{dx}{d\theta} = -4\sin(\theta)\cos(\theta)$

$\frac{dx}{d\theta} = -2\cdot[2\sin(\theta)\cos(\theta)]$

$\frac{dx}{d\theta} = -2\sin(2\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos(2\theta)}{-2\sin(2\theta)} = -\cot(2\theta)$

(a)



$\frac{dy}{d\theta} = 0$

$2\cos(2\theta) = 0$

$\cos(2\theta) = 0$

$2\theta = \frac{\pi}{2}$ or $2\theta = \frac{3\pi}{2}$

$\boxed{\theta = \frac{\pi}{4}}$, or $\boxed{\theta = \frac{3\pi}{4}}$

$\frac{dx}{d\theta} = 0$

$-2\sin(2\theta) = 0$

$\sin(2\theta) = 0$

$2\theta = 0$, or $2\theta = \pi$

$\boxed{\theta = 0}$, or $\boxed{\theta = \frac{\pi}{2}}$

(c)

Horizontal Tangency!

$\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$.

If $\theta = \frac{\pi}{4}$, $r(\frac{\pi}{4}) = \sqrt{2}$

$(\sqrt{2}, \pi/4)$

If $\theta = \frac{3\pi}{4}$, $r(\frac{3\pi}{4}) = -\sqrt{2}$

$(-\sqrt{2}, \frac{3\pi}{4})$

Vertical Tangency; (d)

If $\theta = 0$, $r(0) = 2$

$(2,0)$.

If $\theta = \frac{\pi}{2}$, $r(\frac{\pi}{2}) = 0$

$(0, \frac{\pi}{2})$

(b) $\frac{dy}{dx} = -\cot(2\theta)$

(c) point(*s*) of ***horizontal*** tangency: $(\sqrt{2}, \frac{\pi}{4})$ & $(-\sqrt{2}, \frac{3\pi}{4})$

(d) point(*s*) of ***vertical*** tangency: $(0, \frac{\pi}{2})$ & $(2, 0)$

$\cos^2(\theta)+\sin^2(\theta)=1$, $\cos^2(\theta)=1-\sin^2(\theta)$

15. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta)=1-\sin(\theta)$ over the interval $0\le\theta<2\pi$. **(b)** Find $\frac{dy}{dx}$. **(c)** Find all points of ***horizontal*** tangency. **(d)** Find all points of ***vertical*** tangency. **(e) Label** these points of ***horizontal*** tangency and the points of ***vertical*** tangency on the curve. *(Be careful with your notation, show orientation arrows on your curve, use calculus to demonstrate your results, and show your steps clearly.)*

(b)

$\frac{dr}{d\theta}=\frac{d}{d\theta}[1-\sin(\theta)]=-\cos(\theta)$

$\frac{dy}{d\theta}=r\cos(\theta)+\sin(\theta)\cdot\frac{dr}{d\theta}$

$\frac{dy}{d\theta}=[1-\sin(\theta)]\cdot[\cos(\theta)]+[\sin(\theta)]\cdot[-\cos(\theta)]$

$\frac{dy}{d\theta}=\cos(\theta)-\sin(\theta)\cos(\theta)-\sin(\theta)\cos(\theta)$

$\frac{dy}{d\theta}=\cos(\theta)-2\sin(\theta)\cos(\theta)$

$\frac{dx}{d\theta}=-r\sin(\theta)+\cos(\theta)\cdot\frac{dr}{d\theta}$

$\frac{dx}{d\theta}=-[1-\sin(\theta)][\sin(\theta)]+[\cos(\theta)]\cdot[-\cos(\theta)]$

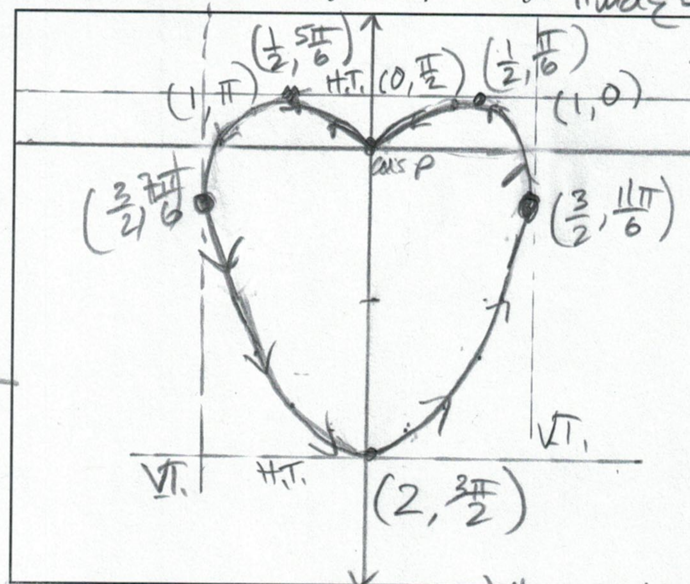
$\frac{dx}{d\theta}=-\sin(\theta)+\sin^2(\theta)-\cos^2(\theta)$

$\frac{dx}{d\theta}=-\sin(\theta)+\sin^2(\theta)-[1-\sin^2(\theta)]$

$\frac{dx}{d\theta}=-\sin(\theta)+\sin^2(\theta)-1+\sin^2(\theta)$

$\frac{dx}{d\theta}=2\sin^2(\theta)-\sin(\theta)-1$

(a) limaçon "cardiod"



$\frac{dy}{d\theta}=0$

$0=\cos(\theta)(1-2\sin(\theta))$

Either $\cos(\theta)=0$, or $1-2\sin(\theta)=0$

$\theta=\frac{\pi}{2}$ or $\theta=\frac{3\pi}{2}$, or $1=2\sin(\theta)$

$\frac{1}{2}=\sin(\theta)$

$\theta=\frac{\pi}{6}$, or $\theta=\frac{5\pi}{6}$

(c) Horizontal Tangency:

$\frac{dy}{d\theta}=0$ & $\frac{dx}{d\theta}\neq 0$.

If $\theta=\frac{\pi}{6}$, $r(\frac{\pi}{6})=\frac{1}{2}$

$(\frac{1}{2},\frac{\pi}{6})$

If $\theta=\frac{5\pi}{6}$, $r(\frac{5\pi}{6})=\frac{1}{2}$

$(\frac{1}{2},\frac{5\pi}{6})$

If $\theta=\frac{3\pi}{2}$, $r(\frac{3\pi}{2})=2$

$(2,\frac{3\pi}{2})$

(b) $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cos(\theta)-2\sin(\theta)\cos(\theta)}{2\sin^2(\theta)-\sin(\theta)-1}$

$\frac{dy}{dx}=\frac{\cos(\theta)(1-2\sin\theta)}{(2\sin(\theta)+1)(\sin(\theta)-1)}$

$\frac{dx}{d\theta}=0$

$0=(2\sin(\theta)+1)(\sin(\theta)-1)$

Either $2\sin(\theta)+1=0$, or $\sin(\theta)-1=0$

$2\sin(\theta)=-1$ $\quad\sin\theta=1$

$\sin(\theta)=-\frac{1}{2}$ $\quad\theta=\frac{\pi}{2}$

$\theta=\frac{7\pi}{6}$, or $\theta=\frac{11\pi}{6}$

(d) Vertical Tangency:

$\frac{dx}{d\theta}=0$ & $\frac{dy}{d\theta}\neq 0$

If $\theta=\frac{7\pi}{6}$, $r(\frac{7\pi}{6})=\frac{3}{2}$

$(\frac{3}{2},\frac{7\pi}{6})$

If $\theta=\frac{11\pi}{6}$, $r(\frac{11\pi}{6})=\frac{3}{2}$

$(\frac{3}{2},\frac{11\pi}{6})$

(b) $\frac{dy}{dx}=\frac{\cos(\theta)(1-2\sin(\theta))}{(2\sin(\theta)+1)(\sin(\theta)-1)}$

(c) point(s) of ***horizontal*** tangency: $(2,\frac{3\pi}{2})$, $(\frac{1}{2},\frac{\pi}{6})$, & $(\frac{1}{2},\frac{5\pi}{6})$

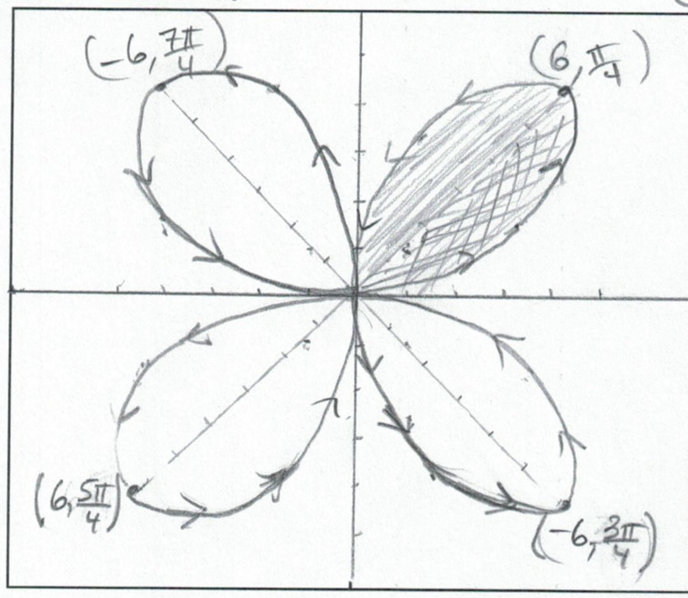
(d) point(s) of ***vertical*** tangency: $(\frac{3}{2},\frac{7\pi}{6})$ & $(\frac{3}{2},\frac{11\pi}{6})$

$$\text{Polar Area} = \frac{1}{2}\int_{\alpha}^{\beta}[r(\theta)]^2\,d\theta \quad , \quad r(\theta)\geq 0 \quad , \quad \frac{1-\cos(2u)}{2} = \sin^2(u)$$

16. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta)=6\sin(2\theta)$ over the interval $0\leq\theta\leq 2\pi$. **(b)** Find the area of one petal of this curve. **(c) Shade** the interior of the petal whose area you are computing. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

(a)



Area of one petal $= 2 \cdot$ (Area of lower half of one petal)

$$= 2\left[\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{4}}[6\sin(2\theta)]^2\,d\theta\right]$$

$$= \int_0^{\frac{\pi}{4}} 36\sin^2(2\theta)\,d\theta$$

$$= 36\int_0^{\frac{\pi}{4}}\sin^2(2\theta)\,d\theta$$

$$= 36\int_0^{\frac{\pi}{4}}\left[\frac{1-\cos(4\theta)}{2}\right]d\theta$$

$$= 18\int_0^{\pi/4}[1-\cos(4\theta)]\,d\theta$$

$$= 18\left[\int_0^{\frac{\pi}{4}}1\,d\theta - \int_0^{\frac{\pi}{4}}\cos(4\theta)\,d\theta\right]$$

$$= 18\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\frac{\pi}{4}}$$

$$= 18\left[\left(\frac{\pi}{4} - \frac{\sin(4(\frac{\pi}{4}))}{4}\right) - \left(0 - \frac{\sin(4(0))}{4}\right)\right]$$

$$= 18\left[\frac{\pi}{4} - \frac{(0)}{4} - 0\right]$$

$$= 18\left(\frac{\pi}{4}\right)$$

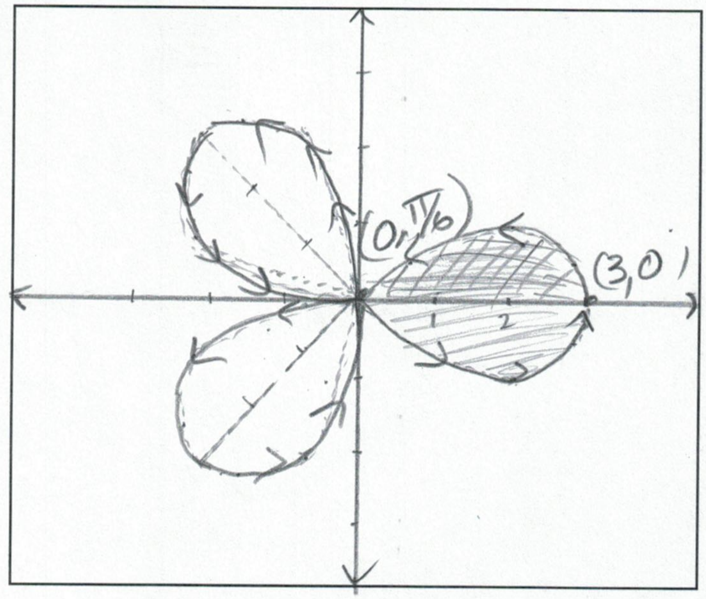
$$= \frac{9\pi}{2}$$

(b) area of one petal of this curve = $\frac{9\pi}{2}$

$$\text{Polar Area} = \frac{1}{2}\cdot\int_{\alpha}^{\beta}[r(\theta)]^2\,d\theta,\quad r(\theta)\geq 0,\quad \frac{1+\cos(2u)}{2}=\cos^2(u)$$

17. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta)=3\cos(3\theta)$ over the interval $0\leq\theta\leq\pi$. **(b)** Find the area of one petal of this curve. **(c)** **Shade** the interior of the petal whose area you are computing. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(a)



b)

$$\text{Area of one petal} = 2\cdot\left(\text{Area of upper half of one petal}\right)$$

$$= 2\cdot\left[\frac{1}{2}\int_0^{\pi/6}[3\cos(3\theta)]^2\,d\theta\right.$$

$$= \int_0^{\frac{\pi}{6}} 9\cos^2(3\theta)\,d\theta$$

$$= 9\int_0^{\pi/6}\cos^2(3\theta)\,d\theta$$

$$= 9\int_0^{\pi/6}\left[\frac{1+\cos(6\theta)}{2}\right]d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6}[1+\cos(6\theta)]\,d\theta$$

$$= \frac{9}{2}\left[\theta+\frac{\sin(6\theta)}{6}\right]_0^{\pi/6}$$

$$= \frac{9}{2}\left[\left(\frac{\pi}{6}+\frac{\sin(6\frac{\pi}{6})}{6}\right)-\left(0+\frac{\sin(6(0))}{6}\right)\right]$$

$$= \frac{9}{2}\left[\frac{\pi}{6}+\frac{0}{6}-0\right]$$

$$= \frac{3\cdot\cancel{3}\cdot\pi}{2\cdot 2\cdot\cancel{3}}$$

$$= \frac{3\pi}{4}$$

(b) area of one petal of this curve = $\frac{3\pi}{4}$

Polar Arclength $= \int_{\alpha}^{\beta} \sqrt{[r(\theta)]^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

18. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta) = 1 + \sin(\theta)$ over the interval $0 \le \theta \le 2\pi$. **(b)** Find the arc length of this curve over the interval $0 \le \theta \le 2\pi$. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

$r(\theta) = 1 + \sin(\theta)$

$\boxed{\frac{dr}{d\theta} = \cos(\theta)}$

Arc Length $= \int_0^{2\pi} \sqrt{[1+\sin(\theta)]^2 + [\cos(\theta)]^2}\, d\theta$

$= 2 \cdot \int_{\pi/2}^{3\pi/2} \sqrt{1 + 2\sin(\theta) + \underbrace{\sin^2(\theta) + \cos^2(\theta)}_{1}}\, d\theta$

$= 2 \int_{\pi/2}^{3\pi/2} \sqrt{1 + 2\sin(\theta) + 1}\, d\theta$

$= 2 \int_{\pi/2}^{3\pi/2} \sqrt{2 + 2\sin(\theta)}\, d\theta$

$= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin(\theta)}\, d\theta$

$= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{\sqrt{1+\sin(\theta)}}{1} \cdot \frac{\sqrt{1-\sin(\theta)}}{\sqrt{1-\sin(\theta)}}\, d\theta$

$= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{\sqrt{1-\sin^2(\theta)}}{\sqrt{1-\sin(\theta)}}\, d\theta$

$= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{\sqrt{\cos^2(\theta)}}{\sqrt{1-\sin(\theta)}}\, d\theta$

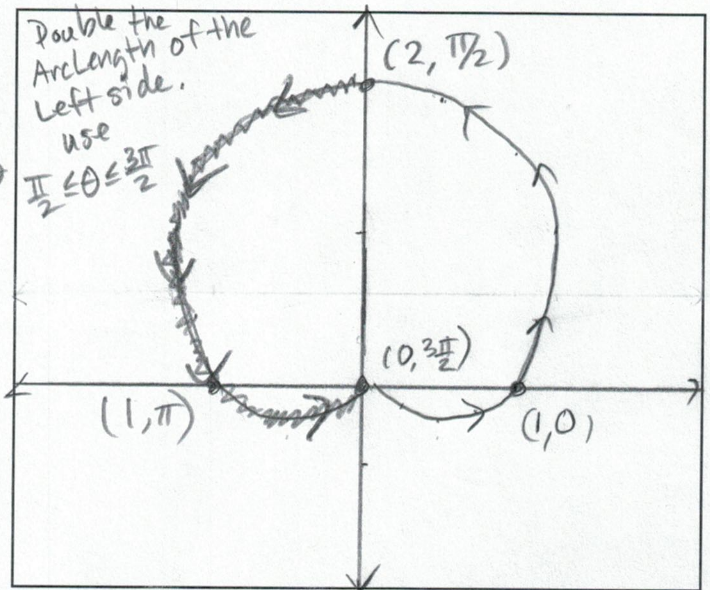
$= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos(\theta)}{\sqrt{1-\sin(\theta)}}\, d\theta$

$= 2\sqrt{2} \int_{u=0}^{u=2} \frac{-\cos(\theta)}{\sqrt{u}} \cdot \left(\frac{du}{-\cos(\theta)}\right)$

$= 2\sqrt{2} \int_0^2 u^{-1/2}\, du$

Improper Integral

(a)



For $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$,
$\cos(\theta) \le 0$.
So, $\sqrt{\cos^2(\theta)} = -\cos(\theta)$

Let $u = 1 - \sin(\theta)$

$\frac{du}{d\theta} = -\cos(\theta)$

$du = \frac{du}{d\theta} \cdot d\theta$

$du = -\cos(\theta)\, d\theta$

$\frac{du}{-\cos(\theta)} = d\theta$

If $\theta = \frac{\pi}{2}$, $u = 1 - \sin(\frac{\pi}{2})$
$u = 1 - 1$
$u = 0$

If $\theta = \frac{3\pi}{2}$, $u = 1 - \sin(\frac{3\pi}{2})$
$u = 1 - (-1)$
$u = 2$

$= 2\sqrt{2} \cdot \lim_{b \to 0^+} \int_b^2 u^{-1/2}\, du$

$= 2\sqrt{2} \cdot \lim_{b \to 0^+} \left[\frac{2}{1} u^{1/2}\right]_b^2$

$= 4\sqrt{2} \cdot \lim_{b \to 0^+} \left[u^{1/2}\right]_b^2$

$= 4\sqrt{2} \cdot \lim_{b \to 0^+} \left[\sqrt{2} - \sqrt{b}\right]$

$= 4\sqrt{2} \cdot [\sqrt{2} - 0]$

$= 4\sqrt{2} \cdot \sqrt{2} = 4 \cdot 2$

$= 8$

(b) arc length of this curve over the interval $0 \le \theta \le 2\pi =$ 8

Polar Arc Length $= \int_{\alpha}^{\beta} \sqrt{[r(\theta)]^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$, $\sin^2(u) = \frac{1-\cos(2u)}{2}$

19. **(a)** Use a graphing utility to graph the curve represented by the following polar equation: $r(\theta) = 2 - 2\cos(\theta)$ over the interval $0 \le \theta \le 2\pi$. **(b)** Find the arc length of this curve over the interval $0 \le \theta \le 2\pi$. *(Be careful with your notation, show orientation arrows on your curve, and show your steps clearly.)*

(b)

$\frac{dr}{d\theta} = -2[-\sin(\theta)]$

$\boxed{\frac{dr}{d\theta} = 2\sin(\theta)}$

Arc Length $= \int_0^{2\pi} \sqrt{[2-2\cos(\theta)]^2 + [2\sin(\theta)]^2}\, d\theta$

$= 2\int_0^{\pi} \sqrt{4 - 8\cos(\theta) + \underbrace{4\cos^2(\theta) + 4\sin^2(\theta)}_{4}}\, d\theta$

$= 2\int_0^{\pi} \sqrt{4 - 8\cos(\theta) + 4}\, d\theta$

$= 2\int_0^{\pi} \sqrt{8 - 8\cos(\theta)}\, d\theta$

$= 2\sqrt{8}\int_0^{\pi} \sqrt{1 - \cos(\theta)}\, d\theta$

$= 2 \cdot 2\sqrt{2} \int_0^{\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= 4\sqrt{2} \cdot \sqrt{2} \int_0^{\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= 4 \cdot 2 \int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$

$= 8 \cdot \left[\frac{2}{1} \cdot \left(-\cos\left(\frac{\theta}{2}\right)\right)\right]_0^{\pi}$

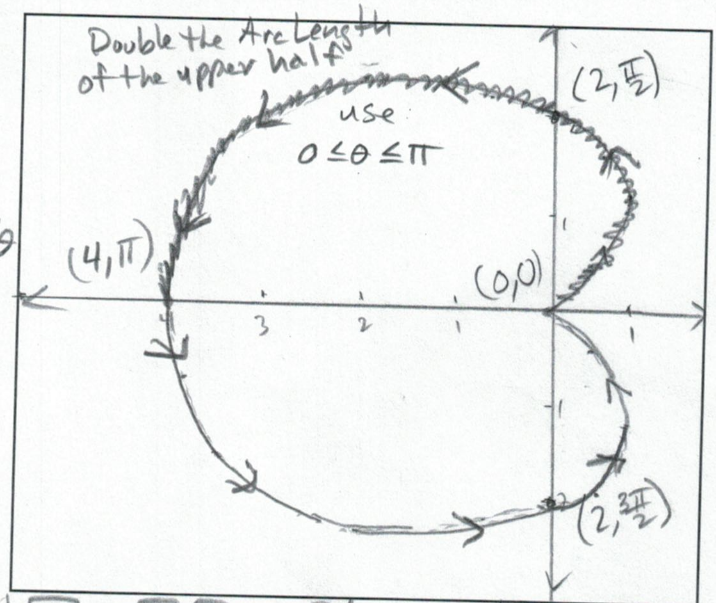
$= -16\left[\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi}$

$= -16\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$

$= -16[0 - 1]$

$= 16$

(a)



$\boxed{\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}}$

for $0 \le \theta \le \pi$ ★★

$\sqrt{\sin^2\left(\frac{\theta}{2}\right)} = \sin\left(\frac{\theta}{2}\right) \ge 0$

$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2}$

$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos(\theta)$

GOOD Reason to use Symmetry and a smaller interval

(b) arc length of this curve over the interval $0 \le \theta \le 2\pi =$ 16