1. **Definition: (Alternating Series)** An alternating series is an infinite series of the form

\[ \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots \]

or

\[ \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + \cdots \]

where \( a_n > 0 \) for all \( n \).

2. **Theorem: Alternating Series Test**

Given an alternating series of the form \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) or \( \sum_{n=1}^{\infty} (-1)^n a_n \), let \( a_n > 0 \) for all \( n \).

If
1. \( a_n \geq a_{n+1} > 0 \) for all \( n \) (i.e. the sequence \( \{a_n\} \) is decreasing for all \( n \)), and
2. \( \lim_{n \to \infty} a_n = 0 \).

Then the alternating series \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) or \( \sum_{n=1}^{\infty} (-1)^n a_n \) converge.

**Note:** Suppose \( \{a_n\} \) is decreasing for all \( n \), but \( \lim_{n \to \infty} a_n \neq 0 \). That does NOT imply that \( \sum_{n=1}^{\infty} (-1)^n a_n \) diverges!!

3. **Theorem: Alternating Series Remainder**

If a convergent alternating series satisfies the condition \( a_n \geq a_{n+1} \) for all \( n \), then the absolute value of the remainder \( R_N \) involved in approximating the sum \( S \) by \( S_N \) is less than or equal to the first neglected term. i.e.

\[ |R_N| = |S - S_N| \leq a_{N+1} \]

4. **Definition: (Absolute Convergence)** We say that the series \( \sum_{n=1}^{\infty} a_n \) converges absolutely if \( \sum_{n=1}^{\infty} |a_n| \) converges. (i.e. the sum is a finite number.)
5. Theorem: Absolute convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

6. Definition: (Conditional Convergence) A series that converges but does NOT absolutely converge is called conditionally convergent. ie. $\sum_{n=1}^{\infty} a_n$ converges conditionally, if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

NOTE: A series can either be

1. $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges. (Absolutely Convergent)
2. $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. (Conditionally Convergent)
3. $\sum_{n=1}^{\infty} a_n$ diverges. (Divergent)