10.2 Plane Curves and Parametric Equations

Definition: If \( f \) and \( g \) are continuous functions of \( t \) on an interval \( I \), then the equations

\[
\begin{align*}
  x &= f(t) \\
  y &= g(t)
\end{align*}
\]

are called parametric equations and \( t \) is called the parameter. The set of points \((x, y)\) obtained as \( t \) varies over \( I \) is the graph of the parametric equations.

Ex #8 Sketch the curve represented by the parametric equations and write the corresponding rectangular equation by eliminating the parameter. \[\begin{align*}
  x &= t^2 + t \\
  y &= t^2 - t
\end{align*}\] \( t \in [-3, 3]\)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.75</td>
<td>-0.25</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1.5</td>
<td>3.75</td>
<td>0.75</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>

Just plug-in values for \( t \) and find \( x \) & \( y \).

Think "clock"
It can be tricky to eliminate $t$ and get an equation with only $x$ & $y$, but let's try.

\[ x = t^2 + t \]

\[ y = t^2 - t \]  \quad \text{solve for } t!

\[ x - y = (t^2 + t) - (t^2 - t) \]
\[ x - y = t^2 + t - t^2 + t \]
\[ x - y = 2t \]

\[ \frac{x - y}{2} = t \]  \quad \text{Substitute into either } x = t^2 + t \text{ or } y = t^2 - t

\[ x = t^2 + t \]

\[ 4x = \left( \frac{x - y}{2} \right)^2 + \left( \frac{x - y}{2} \right) \]

\[ 4x = \frac{(x - y)^2}{4} + \frac{x - y}{2} \]

\[ 4x = (x - y)^2 + 2(x - y) \]
\[ -4x + 4x = x^2 - 2xy + y^2 + 2x - 2y - 4x \]
\[ 0 = x^2 - 2xy + y^2 - 2x - 2y \]  \quad \text{If you want?}?

Restrictions / Range of values for $x$ & $y$.

For $t \in [-3, 3]$

\[ x = t^2 + t \]

\[ x'(t) = 2t + 1 \]

Minimum at $0 = 2t + 1$

\[ t = -\frac{1}{2} \]

\[ x(-\frac{1}{2}) = \frac{3}{4} \]

Max at $t = 3$

\[ (3, 12) \]

\[ x \in [-\frac{1}{2}, 12] \]

Similarly, $y \in [-\frac{1}{2}, 12]$
Example: Use your graphing utility to graph \( x = t^2 \) in parametric mode. \( x = y^2 \)

Example 3

Sketch

\[
\begin{align*}
    x(t) &= 8 \cos \theta & 0 \leq \theta \leq \frac{\pi}{2} \\
y(t) &= 4 \sin \theta
\end{align*}
\]

Then eliminate the parameter to find an equation in \( x \) and \( y \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{4\sqrt{3}}{2} )</td>
<td>2</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \sqrt{2} )</td>
<td>2\sqrt{2}</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>( 2\sqrt{3} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Eliminate \( \theta \):

\[
\begin{align*}
x &= 3 \cos \theta & \quad & y &= 4 \sin \theta \\
\frac{x}{3} &= \cos \theta & \quad & \frac{y}{4} &= \sin \theta \\
\left( \frac{x}{3} \right)^2 &= \cos^2 \theta & \quad & \left( \frac{y}{4} \right)^2 &= \sin^2 \theta
\end{align*}
\]

\[
\cos^2 \theta + \sin^2 \theta = 1
\]

\[
\frac{x^2}{9} + \frac{y^2}{16} = 1
\]

Ellipse: center \((0, 0)\)
Let's modify this situation: 
\[ \begin{align*}
\{ x(t) &= 3\sin(2t) & t & \in \left[ 0, \frac{\pi}{2} \right] \\
y(t) &= 4\cos(2t) 
\end{align*} \]

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$2\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$3\frac{3}{4}$</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$3\frac{3}{4}$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\frac{5\pi}{4}$</td>
<td>$\frac{3}{2}$</td>
<td>$-2\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>0</td>
<td>$-4$</td>
</tr>
</tbody>
</table>

Compare with
\[ \begin{align*}
x(t) &= 3\cos(\frac{\pi}{2} - 2t) \\
y(t) &= 4\sin(\frac{\pi}{2} - 2t) 
\end{align*} \]

\[ \begin{align*}
&x = 3\sin(2t) & y = 4\cos(2t) \\
&x = \sin(2t) & y = 4\cos(2t) \\
&(\frac{x}{4})^2 = \sin^2(2t) & (\frac{y}{4})^2 = \cos^2(2t) \\
&x^2 = 3\sin^2(2t) & y^2 = 3\cos^2(2t) \\
&\sin^2(2t) + \cos^2(2t) = 1 \\
&\frac{x^2}{9} + \frac{y^2}{16} = 1 \\
\end{align*} \]

- we started at a different point
- we moved in the opposite direction
- we moved "fast" at twice as fast
Find a set of parametric equations for the line that passes through (1,4) and (5,2).

Set $t = 0$

When $t = 0$, $x = 1$ and $y = 4$

When $t = 1$, $x = 5$, $y = -2$

$x = 1 + 4t$

$y = 4 - 6t$

Try on graphing calculator

$x(t) = 1 + 4t$

$y(t) = 4 - 6t$

or

$x(t) = 1 + 2t$

$y(t) = 4 - 3t$

or

$x(t) = 1 + 2t^2$

$y(t) = 4 - 3t^2$

$x = 1 + 4t$

$x - 1 = 4t$

$x - 1 = \frac{4t}{1}$

$y = 4 - 6t$

$y = 4 - 6\left(\frac{x-1}{4}\right)$

$y = 4 - \frac{3}{2}(x-1)$

$y = \frac{8}{2} - \frac{3}{2}x + \frac{3}{2}$

$y = \frac{5}{2} - \frac{3}{2}x + \frac{3}{2}$

$y = \frac{5}{2} - \frac{3}{2}x + \frac{11}{2}$