

9.3

The Integral Test + p-Series

Theorem 9.10:

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$, consider

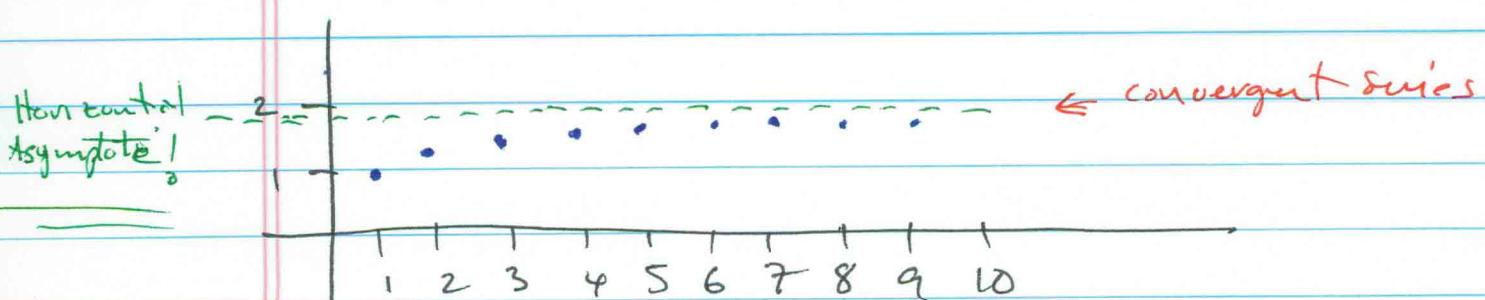
$$\int_1^{\infty} \frac{1}{x^2} dx$$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{b} + 1 \right] \\ &= 1\end{aligned}$$

← the integral converges.

by The Integral Test, (Theorem 9.10),
the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Find the sequence of partial sums using a TI-83??
What did the graph tell you??



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Example: $\sum_{n=1}^{\infty} \frac{1}{n}$, consider $\int_1^{\infty} \frac{1}{x} dx$

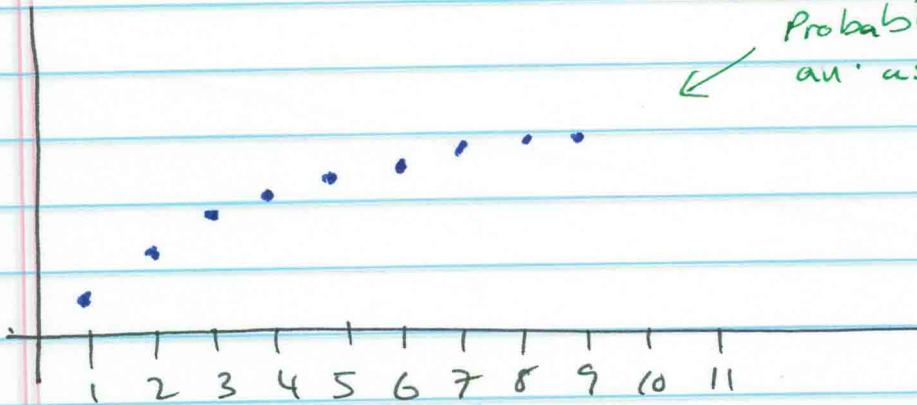
$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} [\ln|x|]_1^b \\ &= \lim_{b \rightarrow \infty} [\ln(b) - \ln(1)]\end{aligned}$$

$= \infty$, \leftarrow The integral diverges.

by The Integral Test, (Theorem 9.10),

the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

General Harmonic Series
 $\sum_{n=1}^{\infty} \frac{1}{an+b}$



Remember Theorem 8.5 for section 8.8

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$$

Theorem 9.11: The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

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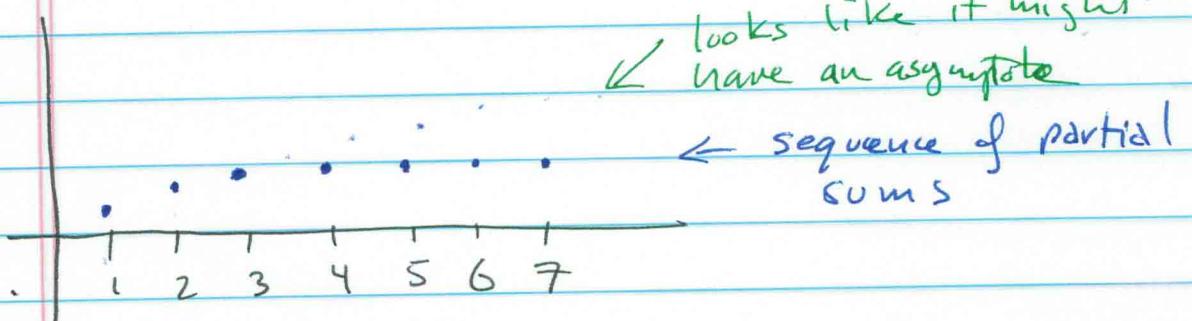
Example 2: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, consider $\int_1^{\infty} \frac{1}{x^2+1} dx$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2+1} dx \\ &= \lim_{b \rightarrow \infty} [\arctan(x)]_1^b \\ &= \lim_{b \rightarrow \infty} [\arctan(b) - \arctan(1)] \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4}\end{aligned}$$

\leftarrow The integral converges.

by the Integral Test, (Theorem 9.10),

the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.



#4 Consider $\sum_{n=1}^{\infty} n e^{-\frac{n}{2}}$ using the Integral Test:

$$\begin{aligned}\int_1^{\infty} x e^{-\frac{x}{2}} dx &= \lim_{b \rightarrow \infty} \int_1^b x e^{-\frac{x}{2}} dx \\ &= \lim_{b \rightarrow \infty} \left[(x)(-2e^{-\frac{x}{2}}) \right]_1^b - \int_1^b (-2e^{-\frac{x}{2}})(dx)\end{aligned}$$

use "parts"

$$\int u dv = uv - \int v du$$

$$\text{let } u = x$$

$$\begin{aligned}\frac{du}{dx} &= 1 \\ du &= dx\end{aligned}$$

$$dv = e^{-\frac{x}{2}} dx$$

$$v = \int e^{-\frac{x}{2}} dx$$

$$v = \int e^{-\frac{x}{2}} (-2 dz)$$

$$v = -2 e^{-\frac{x}{2}}$$

$$\begin{cases} \text{let} \\ z = -\frac{x}{2} \end{cases}$$

$$\begin{cases} \frac{dz}{dx} = -\frac{1}{2} \\ -2 dz = dx \end{cases}$$

$$-2 dz = dx$$

9.3) #4 cont'd

$$\begin{aligned}
 &= \lim_{b \rightarrow \infty} \left[-2x e^{-\frac{x}{2}} \Big|_1^b + 2 \int_1^b e^{-\frac{x}{2}} dx \right] \\
 &= \lim_{b \rightarrow \infty} \left[-2x e^{-\frac{x}{2}} \Big|_1^b + 2 \left[-2e^{-\frac{x}{2}} \right]_1^b \right] \\
 &= \lim_{b \rightarrow \infty} \left[-2x e^{-\frac{x}{2}} - 4e^{-\frac{x}{2}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left(-2b e^{-\frac{b}{2}} - 4e^{-\frac{b}{2}} \right) - \left(-2(1)e^{-\frac{1}{2}} - 4e^{-\frac{1}{2}} \right) \\
 &= \lim_{b \rightarrow \infty} \left(\frac{-2b}{e^{b/2}} - \frac{4}{e^{b/2}} + \frac{2}{e^{1/2}} + \frac{4}{e^{1/2}} \right) \\
 &= \lim_{b \rightarrow \infty} \frac{-2b}{e^{b/2}} - 4 \cdot \lim_{b \rightarrow \infty} \frac{1}{e^{b/2}} + \frac{6}{\sqrt{e}} \\
 &= \lim_{b \rightarrow \infty} \frac{\frac{d}{db}(-2b)}{\frac{d}{db}(e^{b/2})} - 4 \cdot 0 + \frac{6}{\sqrt{e}} \\
 &= \lim_{b \rightarrow \infty} \frac{-2}{\frac{1}{2} \cdot e^{b/2}} + \frac{6}{\sqrt{e}} \\
 &= 0 + \frac{6}{\sqrt{e}}
 \end{aligned}$$

lim
 b → ∞ $\frac{-2b}{e^{b/2}}$
 = $\frac{-\infty}{\infty}$
 ∴
 use
 L'Hopital's
 Rule

$= \frac{6}{\sqrt{e}} \quad \leftarrow \text{the integral converges}$

by the Integral test
 the series $\sum_{n=1}^{\infty} n e^{-\frac{n}{2}}$ converges.