Theorem 9.14  Alternating Series Test

Let \( a_n > 0 \). The alternating series

\[
\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n
\]

converge if the following two conditions are met:

1. \( \lim_{n \to \infty} a_n = 0 \), AND
2. \( a_{n+1} \leq a_n \), for all \( n \)

Example: Consider \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)

Try to satisfy both criteria for Theorem 9.14:

1. \( \lim_{n \to \infty} a_n = 0 \)

   Evaluate: \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \)

2. Show \( a_{n+1} \leq a_n \), for all \( n \)

   \[
   \frac{1}{n+1} \leq \frac{1}{n}
   \]

   So, \( a_{n+1} \leq a_n \)

   Therefore, by Theorem 9.14, the Alternating Series Test, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges.
Consider \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \]

Try to satisfy both criteria for Theorem 9.14

1. \[ \lim_{n \to \infty} a_n = 0 \]

   
   \[ = \lim_{n \to \infty} \frac{n}{n^2+1} \cdot \left( \frac{1}{\frac{n^2}{n^2}} \right) \]
   \[ = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} \]
   \[ = 0 \]

2. Show \[ a_{n+1} \leq a_n \] for all \( n \geq 1 \)

   
   Show: \[ \frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1} \]

\[ n^3 + 2n^2 + n + 1 \geq n^3 + n^2 + n + 1 \]

\[ n^3 + 2n^2 + 2n + 2 \geq n^3 + n^2 + n + 1 \]

\[ n(n^2 + 2n + 2) \geq (n^2 + 1)(n + 1) \]

\[ \frac{n}{n^2 + 1} \geq \frac{n + 1}{n^2 + 2n + 2} \]

\[ \frac{n}{n^2 + 1} \geq \frac{n + 1}{(n^2 + 2n + 1) + 1} \]

\[ \frac{n}{n^2 + 1} \geq \frac{n + 1}{(n + 1)^2 + 1} \]

\[ a_n \geq a_{n+1} \]
Therefore, by Theorem 9.14, the alternating series test, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} \) converges.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} \]

Sequence of partial sums

looks like ???

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} \]

#18 Consider \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} \)

Use \( n^{th} \)-Term Test for Divergence ?

If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. If \( \lim_{n \to \infty} a_n \neq 0 \), the 1st criterion would fail.

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{n^2 + 5} \]

\[ = \lim_{n \to \infty} \left( \frac{n^2}{n^2 + 5} \right) \left( \frac{1/5}{1/5} \right) \]

\[ = \lim_{n \to \infty} \frac{1}{1 + \frac{5}{n^2}} \]

\[ = \frac{1}{1} \]

\[ \lim_{n \to \infty} a_n = 1 \]

Therefore, by the \( n^{th} \)-Term Test, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} \) diverges.
Example: Consider \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \ldots \)

Earlier, we saw that this series was convergent. Can we approximate its sum?

\[
\begin{align*}
S_{10} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \\
&= \frac{1627}{2520} \\
&\approx 0.64563
\end{align*}
\]

We can say \( S_{10} \approx S \), where \( S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \).

But, what is the error in our approximation?

\[
S = S_N + R_N \quad \text{and} \quad R_N = S - S_N
\]

where \( R_N \) is the remainder.

Theorem 9.15: Alternating Series Remainder

If a convergent alternating series satisfies the condition \( a_n \leq a_{n+1} \), then the absolute value of the remainder \( R_N \) involved in approximating the sum \( S \) by \( S_N \) is less than (or equal to) the first neglected term.

That is,

\[
| S - S_N | = | R_N | \leq | a_{n+1} |
\]

Proof: Let \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) be a convergent alternating series satisfying \( \lim_{n \to \infty} a_n = 0 \) and \( a_n \leq a_{n+1} \).
9.5 \text{ } 10 + 9.15 \text{ } \text{an}^{10}

Consider \( R_n = S - S_n \)

\[ R_n = \sum_{n=1}^{N} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n \]

\[ = (-1)^{n+1} + (-1)^{n+2} a_{n+2} + (-1)^{n+3} a_{n+3} + (-1)^{n+4} a_{n+4} + \ldots \]

\[ = (-1)^{n+2} \left[ a_{n+1} - (-1)^n a_{n+2} + (-1)^n a_{n+3} - (-1)^n a_{n+4} + \ldots \right] \]

\[ |R_n| = |(-1)^{n+2} \left[ a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \ldots \right]| \]

\[ |R_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \ldots \]

\[ = a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) + \ldots \leq a_{n+1} \]

So,

\[ |R_n| \leq a_{n+1} \]

**Back to** \( S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \)

With \( S_{10} \approx S \)

And \( S < 0.64563 \)

We'll have \( R_{10} = \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} + \ldots \)

So

\[ R_{10} = \frac{1}{11} - \left( \frac{1}{12} - \frac{1}{13} \right) - \left( \frac{1}{14} - \frac{1}{15} \right) - \ldots \leq \frac{1}{11} \]

This means that \( S_{10} \) is within \( \frac{1}{11} \) of \( S \).

Or,

\[ |S - S_{10}| \leq \frac{1}{11} \]

\[ -\frac{1}{11} \leq S - 0.64563 \leq \frac{1}{11} \]

\[ 0.64563 - \frac{1}{11} \leq S \leq 0.64563 + \frac{1}{11} \]

\[ 0.64563 \leq S \leq 0.73654 \]
(a) Use Theorem 9.15 to determine the number of terms required to approximate the sum of the convergent series with an error of less than 0.001. (b) Use a graphing utility to approximate the sum of the series with an error less than 0.001.

\[ \sum_{n=0}^{\infty} \frac{61^n}{(2n)!} = \cos(1) \]

\[ \cos(1) \approx 0.5403023059 \]

Consider

(a) The approximation error is the remainder. By Theorem 9.15 we have

\[ |R_N| \leq a_{N+1} \] \quad \text{with} \quad a_n = \frac{1}{(2n)!} \]

So

\[ |R_N| \leq a_{N+1} = \frac{1}{[2(N+1)]!} < 0.001 \]

If we solve for \( N \), we can find the number of terms we need to know \( S_N \) is within 0.001 of \( S \).

Solve

\[ \frac{1}{(2N+2)!} < \frac{1}{1000} \]

for \( N \):

\[ N = 1, \quad \frac{1}{(2+2)!} = \frac{1}{4!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{24} \]

\[ N = 2, \quad \frac{1}{(4+2)!} = \frac{1}{6!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{1}{720} \]

\[ N = 3, \quad \frac{1}{(6+2)!} = \frac{1}{8!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} = \frac{1}{40320} \]

\[ S_2 < 0.001 \text{ when } N \geq 3 \]

Since we start with \( n=0 \) in \( \sum_{n=0}^{\infty} \frac{61^n}{(2n)!} \), we'll use 4 terms when \( N = 3 \) for \( S_3 \).
\[ S_3 = \frac{(-1)^0}{2(0)!} + \frac{(-1)^1}{2(1)!} + \frac{(-1)^2}{2(2)!} + \frac{(-1)^3}{2(3)!} \]
\[ S_3 = \frac{1}{0!} + \frac{(-1)}{2!} + \frac{1}{4!} + \frac{(-1)}{6!} \]
\[ S_3 = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \]
\[ S_3 = \frac{389}{720} \]

(b) \( S_3 = 0.54027 \), or \( S_3 \approx 0.5403 \)

Theorem 9.16: Absolute Convergence

If the series \( \sum |a_n| \) converges, then the series \( \sum a_n \) converges.

**Proof:** If \( \sum |a_n| \) converges, then \( \sum 2|a_n| \) converges.

For all \( n \), we can see that
\[ 0 \leq a_n + |a_n| \leq 2|a_n| \]

So, by Theorem 9.12, the Direct Comparison Test,
\[ \sum (a_n + |a_n|) \] converges,
\[ \sum 2|a_n| \] converges.

By writing \( \sum a_n = \sum (a_n + |a_n|) - \sum |a_n| \)

We can see that the series \( \sum a_n \) converges, since both \( \sum (a_n + |a_n|) \) and \( \sum |a_n| \) are convergent series.

**Definitions:**

1. \( \sum a_n \) is absolutely convergent if \( \sum |a_n| \) converges.
2. \( \sum a_n \) is conditionally convergent if \( \sum a_n \) converges but \( \sum |a_n| \) diverges.
Consider \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}} \)

1. \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\sqrt{n+4}} = 0 \checkmark \)

2. Show \( a_{n+1} \leq a_n \)

\[ a_{n+1} = \frac{1}{\sqrt{(n+1)+4}}, \quad a_n = \frac{1}{\sqrt{n+4}} \]

\[ (n+1) \geq n \]

\[ (n+1) + 4 \geq n + 4 \]

\[ \frac{1}{\sqrt{(n+1)+4}} \leq \frac{1}{\sqrt{n+4}} \]

So, \( a_{n+1} \leq a_n \) \( \checkmark \)

By the Alternating Series Test, \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}} \) converges.

Does \( \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right| \) converge?

\[ \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} \]

Try to show \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} \) is divergent. Use the Limit Comparison test with a \( p \)-series, \( p = \frac{1}{2} \)

Sequence of partial sums for \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}} \) looks like \( \frac{8}{10} \)

Sequence of partial sums for \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} \) looks like divergence.
\[
\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.
\]

Divergent series \( p = \frac{1}{2} \)

\[
\frac{1}{\sqrt{n+3}} \geq 0 \quad \text{and} \quad 0 \leq \frac{1}{\sqrt{n}}
\]

Evaluate: \(\lim_{n \to \infty} \left( \frac{\sqrt{n}}{\sqrt{n+3}} \right) = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} \cdot \sqrt{1 + \frac{3}{n}}} = \lim_{n \to \infty} \frac{\sqrt{n}}{n \cdot \sqrt{1 + \frac{3}{n}}} = \lim_{n \to \infty} \frac{1}{\sqrt{1 + \frac{3}{n}}} = 1 \)

Finite and positive!

By the Limit Comparison Test, both series diverge.

Since \(\sum_{n=0}^{\infty} \frac{(n+1)^n}{n^{n+4}}\) diverges,

we can say that \(\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}\) diverges conditionally.

Consider \(\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}\) looks like \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\) convergent \(p\)-series

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0
\]
2. show \( a_{n+1} \leq a_n \)

\[
an_{n+1} = \frac{1}{(n+1)(\sqrt{n+1})}, \quad a_n = \frac{1}{n^{1/2}}
\]

\[
n+1 \geq n
\]

\[
\frac{1}{(n+1)^{1/2}(n+1)} \leq \frac{1}{n^{1/2}}
\]

So, \( a_{n+1} \leq a_n \). By the Alternating Series Test, \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}} \) converges.

Does \( \sum_{n=1}^{\infty} \frac{|(-1)^n|}{n^{1/2}} \) converge? Yes.

\[
\sum_{n=1}^{\infty} \frac{|(-1)^n|}{n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}
\]

which is a convergent \( p \)-series with \( p = \frac{3}{2} \).

This means that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}} \) converges absolutely.