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# Monotone difference approximations for the simulation of clarifier-thickener units

**R. Bürger**[1], **K.H. Karlsen**[2], **N.H. Risebro**[3], **J.D. Towers**[4]

[1] Institut für Angewandte Analysis und numerische Simulation, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany, E-mail: buerger@mathematik.uni-stuttgart.de
[2] Department of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway, E-mail: kennethk@mi.uib.no
[3] Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N-0316 Oslo, Norway, E-mail: nilshr@math.uio.no
[4] MiraCosta College, 3333 Manchester Avenue, Cardiff-by-the-Sea, CA 92007-1516, USA, E-mail: jtowers@cts.com

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**Abstract.** Clarifier-thickener units treating ideal suspensions can be modeled as an initial-value problem for a nonconvex scalar conservation law whose flux depends on a vector of discontinuous parameters. This problem can be treated by the well-known Engquist-Osher scheme if the discontinuous paremeters are discretized on a grid staggered against that of the conserved variable. We prove convergence of this scheme to a weak solution of the problem and illustrate its application to the clarifier-thickener setup by a numerical example.

**Key words** conservation laws, discontinuous flux, sedimentation, monotone difference scheme

## 1 Introduction

The extension of Kynch's kinematic model of sedimentation of ideal suspensions [5,12] to continuous clarification-thickening processes leads to a nonconvex scalar conservation law with discontinuous flux for the local solids concentration as a function of depth and time. Several authors have extended such setups to non-cylindrical vessels [6,7,17] under the assumption that the horizontal cross-sectional area $S$ is a smooth function of the depth $x$. In this paper, we assume that $S(x)$ is only piecewise continuous, which means that the diameter jumps. In this case the currently available mathematical and numerical theory breaks down. We show that these difficulties can be overcome by employing an easily implemented extension of the well-known Engquist-Osher scheme [8]. In particular, it is proved that the scheme converges towards a weak solution of the problem.

The model studied here is very similar to problems of traffic flow with abruptly changing road conditions [13] and of two-phase flow in porous media with jumps in permeability [9]. Thus, the numerical scheme studied herein is applicable to a variety of models.

*Send offprint requests to*: R. Bürger

The remainder of the paper is organized as follows. In Section 2 we outline the mathematical model. The numerical method is introduced in Section 3 and applied to illustrate the mathematical model in Section 4. The main result, the convergence of the scheme, is stated in Theorem 5.1 at the beginning of Section 5 and subsequently proved by a sequence of lemmas.

## 2 Mathematical model

We consider a settling vessel with a variable horizontal cross-sectional area $S(x)$, where $-1 \leq x \leq 1$ is the dimensionless depth variable, see Figure 4.1. We let $x$ increase downwards, and assume that $x=-1$ is the overflow level, $x=0$ corresponds to the feed level and $x=1$ is the discharge level. The unknown volumetric solids concentration $u$ is assumed to depend on height only, $u=u(x,t)$. Then the conservation of mass equation for the solids and the fluid is given by

$$S(x)u_t + \big(S(x)uv_{\mathrm{s}}\big)_x = 0, \tag{2.1}$$

$$-S(x)u_t + \big(S(x)(1-u)v_{\mathrm{f}}\big)_x = 0, \tag{2.2}$$

respectively, where $t$ is time and $v_{\mathrm{s}}$ and $v_{\mathrm{f}}$ are the solids and the fluid phase velocities. The mixture flux, that is the volume average flow velocity weighted with $S(x)$, is given by $Q(x,t) := S(x)(uv_{\mathrm{s}} + (1-u)v_{\mathrm{f}})$. The sum of (2.1) and (2.2) produces the continuity equation of the mixture, $Q_x = 0$ for $-1 < x < 0$ and $0 \leq x \leq 1$ and $t > 0$, which implies that $Q(\cdot,t)$ is constant as a function of $x$. Since $Q$ only suffers a jump across the feed source level $x=0$, we get for $t>0$

$$Q(x,t) = \begin{cases} Q(-1,t) = Q_{\mathrm{L}}(t) & \text{for } -1 \leq x < 0, \\ Q(1,t) = Q_{\mathrm{R}}(t) & \text{for } 0 < x \leq 1, \end{cases} \tag{2.3}$$

where $Q_{\mathrm{L}}(t) \leq 0$ and $Q_{\mathrm{R}}(t) \geq 0$ are the signed volumetric suspension overflow and discharge rates, respectively, prescribed by the control of the suspension volume flows

at $x = -1$ and $x = 1$. Note that we here identify a downwards flow with a flow to the right.

We let (2.3) replace (2.2) and rewrite (2.1) in terms of $Q(x,t)$ and the solid-fluid relative velocity or slip velocity $v_\mathrm{r} := v_\mathrm{s} - v_\mathrm{f}$, for which a constitutive equation will be formulated. Observing that $uv_\mathrm{s} = [Q(x,t)u/S(x)] + u(1-u)v_\mathrm{r}$, and assuming for a moment that no solids enter the vessel at $x = 0$, we obtain from (2.1)

$$S(x)u_t + \big(Q(x,t)u + S(x)u(1-u)v_\mathrm{r}\big)_x = 0. \qquad (2.4)$$

The well-known kinematic sedimentation theory [5] is based on the assumption that $v_\mathrm{r}$ is a function of $u$ only, $v_\mathrm{r} = v_\mathrm{r}(u)$. We here express $v_\mathrm{r}$ in terms of the so-called Kynch batch flux density function $h$ as $v_\mathrm{r}(u) = h(u)/(u(1-u))$, such that (2.4) takes the form

$$S(x)u_t + \big(Q(x,t)u + S(x)h(u)\big)_x = 0. \qquad (2.5)$$

The function $h$ is assumed to satisfy $h(u) = 0$ for $u \leq 0$ or $u \geq u_{\max}$, where $u_{\max}$ is the maximum solids concentration, $h(u) > 0$ for $0 < u < u_{\max}$, $h'(0) > 0$ and $h'(u_{\max}) \leq 0$. For simplicity we assume (possibly after rescaling $u$) that $u_{\max} = 1$. A common function is [5]

$$h(u) = v_\infty u(1-u)^N, \quad v_\infty > 0,\ N \geq 1, \qquad (2.6)$$

where $v_\infty > 0$ is the appropriately scaled settling velocity of a single particle in an unbounded pure fluid. For our analysis we assume that $h \in C^2[0,1]$, and that $|h''| > 0$ except for finitely many inflection points.

We assume that $0 < S_{\min} \leq S(x) \leq S_{\max}$ for $x \in \mathbb{R}$ and $\mathrm{TV}_x(S) < \infty$, and that $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $0 \leq u_0(x) \leq 1$ for $x \in \mathbb{R}$. To simplify the problem further, we assume $Q_\mathrm{L}(t) = Q_\mathrm{L} = \mathrm{const.}$ and $Q_\mathrm{R}(t) = Q_\mathrm{R} = \mathrm{const.}$ As in [1,2], we assume that at $x = 1$, the composite flux $Q_\mathrm{R}u + S(x)h(u)$ is changed to the discharge transport flux $Q_\mathrm{R}u$, that at $x = -1$, the composite flux $Q_\mathrm{L}u + S(x)h(u)$ is changed to the overflow transport flux $Q_\mathrm{R}u$, and that at $x = 0$, a feed source is located. Physically, we assume that the underflow and overflow mixtures are transported away from the unit through pipes of constant small cross-sectional areas $S_\mathrm{R}$ and $S_\mathrm{L}$ with $S_{\min} \leq S_\mathrm{L}, S_\mathrm{R} \leq S_{\max}$, respectively, and that the solids and the fluid are transported away at the same speed, such that the relative velocity $v_\mathrm{r}(u)$ and thus $h(u)$ is not present for $|x| > 1$. A clarifier-thickener unit with constant cross-sectional area $S_0$ [1,2] can be modeled by the equation $u_t + \hat{g}(u,x)_x = 0$ with the composite flux

$$\hat{g}(u,x) := \begin{cases} q_\mathrm{L}u & \text{for } x < -1, \\ q_\mathrm{L}u + h(u) & \text{for } -1 < x < 0, \\ q_\mathrm{R}u + h(u) + (q_\mathrm{L} - q_\mathrm{R})u_\mathrm{F} & \text{for } 0 < x < 1, \\ q_\mathrm{R}u + (q_\mathrm{L} - q_\mathrm{R})u_\mathrm{F} & \text{for } x > 1, \end{cases}$$

where $q_\mathrm{L} = Q_\mathrm{L}/S_0 \leq 0$ and $q_\mathrm{R} = Q_\mathrm{R}/S_0 \geq 0$ are the volume overflow and discharge rates, respectively, divided by the cross-sectional area, and $u_\mathrm{F} \in [0, u_{\max}]$ is the concentration of the suspension fed into the unit through the singular source at $x = 0$ at the volumetric rate $Q_\mathrm{F} = Q_\mathrm{R} - Q_\mathrm{L}$. Considering now that $S(x)$ is allowed to vary, we obtain the conservation law

$$S(x)u_t + g(u,x)_x = 0, \quad x \in \mathbb{R},\ t > 0, \qquad (2.7)$$

$$g(u,x) := \begin{cases} Q_\mathrm{L}(u - u_\mathrm{F}) & \text{for } x < -1, \\ Q_\mathrm{L}(u - u_\mathrm{F}) + S(x)h(u) & \text{for } -1 < x < 0, \\ Q_\mathrm{R}(u - u_\mathrm{F}) + S(x)h(u) & \text{for } 0 < x < 1, \\ Q_\mathrm{R}(u - u_\mathrm{F}) & \text{for } x > 1, \end{cases} \qquad (2.8)$$

together with the initial condition

$$u(x,0) = u_0(x) \in [0, u_{\max}], \quad x \in \mathbb{R}. \qquad (2.9)$$

Independently of the smoothness of $g(u,\cdot)$ and $u_0$, solutions of (2.7)–(2.9) generally develop discontinuities, and so weak solutions must be sought.

**Definition 2.1.** *A* weak solution *of the initial value problem* (2.7)–(2.9) *is a bounded function* $u(x,t)$ *satisfying, for all test functions* $\phi \in \mathcal{D}(\mathbb{R} \times (0,T))$ *with* $\phi|_{t=T} = 0$,

$$\int_0^T \int_\mathbb{R} \big(S(x)u\phi_t + g(u,x)\phi_x\big)\,dxdt + \int_\mathbb{R} S(x)u_0(x)\phi(x,0)dx = 0.$$

## 3 Numerical method

We define the composite flux $f(\boldsymbol{\gamma}(x), u) := g(u,x)$, where the parameter vector $\boldsymbol{\gamma}(x) := (\gamma_1(x), \gamma_2(x))$ is given by

$$\gamma_1(x) := S(x)\chi_{[-1,1]}(x), \quad \gamma_2(x) := \begin{cases} Q_\mathrm{L} & \text{for } x < 0, \\ Q_\mathrm{R} & \text{for } x > 0, \end{cases}$$

such that $f(\boldsymbol{\gamma}, u) = \gamma_2(u - u_\mathrm{F}) + \gamma_1 h(u)$. Thus, the initial value problem for Equation (2.7) reads

$$S(x)u_t + f\big(\boldsymbol{\gamma}(x), u\big)_x = 0, \quad u(x,0) = u_0(x). \qquad (3.1)$$

To define a numerical method for (3.1), we choose $\Delta x > 0$, set $x_j := j\Delta x$, and discretize the parameter vector $\boldsymbol{\gamma}$, the initial data, and the cross sectional area by

$$\boldsymbol{\gamma}_{j+\frac{1}{2}} := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \boldsymbol{\gamma}(x)\,dx,\ U_j^0 := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\,dx,$$
$$S_j := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(x)\,dx.$$

For $n > 0$ we define the approximations according to the explicit marching formula

$$U_j^{n+1} = U_j^n - \lambda_j \Delta_- f^{\mathrm{EO}}\big(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n\big), \qquad (3.2)$$

where $\lambda_j := \Delta t/(S_j \Delta x)$, $\Delta_- V_j := V_j - V_{j-1}$, and

$$f^{\mathrm{EO}}(\boldsymbol{\gamma}, v, u) := \frac{1}{2}\left[f(\boldsymbol{\gamma}, u) + f(\boldsymbol{\gamma}, v) - \int_u^v |f_u(\boldsymbol{\gamma}, w)|dw\right]$$

is the Engquist-Osher flux. Let $t_n = n\Delta t$ and let $\chi^n$ denote the characteristic function of $[t_n, t_{n+1})$ and $\chi_j$ the characteristic function of $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. We then define

$$u^\Delta(x,t) := \sum_{n \geq 0}\sum_{j \in \mathbb{Z}} U_j^n \chi_j(x)\chi^n(t),$$
$$\boldsymbol{\gamma}^\Delta(x) := \sum_{j \in \mathbb{Z}} \boldsymbol{\gamma}_{j+\frac{1}{2}}\chi_{j+\frac{1}{2}}(x).$$

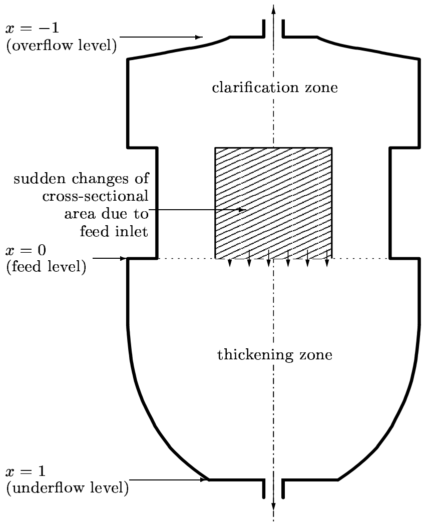


**Fig. 4.1.** Cross-section of a clarifier-thickener unit having the cross-sectional area function $S_1(x)$.

# 4 Numerical example

To illustrate the model of continuous sedimentation in clarifier-thickener units, we have drawn in Figure 4.1 the cross-section of a (hypothetical) clarifier-thickener with the cross-sectional area function $S(x) = S_1(x)$ given by

$$S_1(x) = \begin{cases} 0.1 + 90(x+1)^2 & \text{for } -1 \le x \le -0.9, \\ 1 & \text{for } -0.9 < x \le -0.5, \\ 0.5 & \text{for } -0.5 < x \le 0, \\ 1 & \text{for } 0 < x \le 0.3, \\ 1 - \frac{0.8}{0.49}(x-0.3)^2 & \text{for } 0.3 < x \le 1, \\ S_\mathrm{L} = S_\mathrm{R} = 0.01 & \text{otherwise.} \end{cases}$$

In Figure 4.2 we show a numerical solution using the scheme outlined in Section 3 with $u_0 \equiv 0$, $Q_\mathrm{L} = -0.1$, $Q_\mathrm{R} = 0.1$ and $u_\mathrm{F} = 0.4$. The function $h(u)$ is given by (2.6) with $v_\infty = 1$ and $N = 5$. The numerical parameters are $\Delta x = 1/100$ and $\lambda = 0.05$. Note that the visual time step in Figure 4.2 is larger than the computational.

The simulation shows that the solids initially settle exclusively into the thickening zone ($x > 0$), accumulate there and form a rising sediment. This sediment layer breaks through the feed level and the particles entering the clarification zone start to produce an overflow of clarified suspension. The shape of the concentration surface in the clarification zone reflects the vessel geometry, and that there the solution becomes stationary. Furthermore, the concentration leaving the unit increases at $x = 1$ and decreases at $x = -1$ discontinuously. The system converges to a steady state of simultaneous clarification and thickening. We refer to [2–4] for further examples.

# 5 Convergence of the numerical scheme

To state the main result, we let $f_u^+(w) := \max\{f_u(w), 0\}$ and $f_u^-(w) := \min\{f_u(w), 0\}$, and assume that the ratio

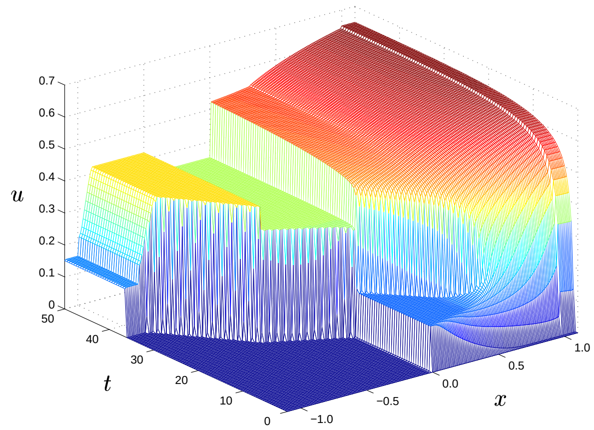


**Fig. 4.2.** Simulation of filling up and attaining a steady state of the clarifier-thickener unit drawn in Figure 4.1.

$\Delta t/\Delta x$ is chosen so that the CFL condition

$$\lambda_j \max_{U \in (0,1)} \left| f_u^+(\gamma_{j+\frac{1}{2}}, U) - f_u^-(\gamma_{j-\frac{1}{2}}, U) \right| \le 1 \quad \forall j \in \mathbb{Z} \tag{5.1}$$

is satisfied. The purpose of this section is to outline the proof of the following theorem.

**Theorem 5.1.** *Let the approximate solution $u^\Delta$ be given by the scheme of Section 3. Let $\Delta \to 0$ with $\Delta t/\Delta x =$ const. and the CFL condition* (5.1) *be satisfied. Assume that* $\mathrm{TV}(u_0) < \infty$. *Then there exists a subsequence of $\{u^\Delta\}$, still denoted by $\{u^\Delta\}$, and a function $u$ such that $u^\Delta \to u$ in $L^1_\mathrm{loc}(\mathbb{R} \times (0,T))$ and a.e. in $\mathbb{R} \times (0,T)$. The limit function $u$ is a weak solution in the sense of Def. 2.1.*

The strategy of the proof of Theorem 5.1 is similar to the proofs found in [15,16], but some details are more complicated here. The proof is divided into a series of lemmas as follows. Lemma 5.1 states that the discrete solutions $u^\Delta$ remain uniformly bounded, that the scheme is monotone, and that the approximate solutions are $L^1$ Lipschitz continuous in time. Lemmas 5.2 and 5.3 lead to a pair of discrete entropy inequalities parametrized by the real parameter of the well-known Kružkov entropy pair [11]. As in [1,4,16], we establish compactness of the sequence of approximate solutions by application of a singular mapping $\Psi$, which is frequently referred to as *Temple functional* [14] and is defined here as

$$\Psi(\gamma, u) := \int_0^u \left| f_u(\gamma, w) \right| dw. \tag{5.2}$$

In Lemma 5.4 we state three basic monotonicity and continuity properties of this functional, which in particular imply its invertibility. Proceeding in a similar way as in [15,16], we decompose $\Psi$ into a finite number of components with support limited to maximal segments on which $f$ is a monotone function of $u$. The construction

of this decomposition becomes rather involved here since the flux $h(u)$ is non-convex and a *vector* $\boldsymbol{\gamma}$ of discontinuous parameters is considered, and is cast into Lemma 5.5 and its proof. Utilizing the entropy inequalities derived in Lemma 5.3, we then prove in Lemmas 5.6 and 5.7 that each of these components of $\Psi$ has bounded variation.

For $z^{\Delta} := \Psi(\boldsymbol{\gamma}^{\Delta}, u^{\Delta})$, this sequence of lemmas leads to the inequalities

$$|z^{\Delta}(x,t)| \leq \max_{\boldsymbol{\gamma}\in\Gamma} \Psi(\boldsymbol{\gamma}, 1), \tag{5.3}$$

$$\left\| z^{\Delta}(\cdot, t_1) - z^{\Delta}(\cdot, t_2)\right\|_{L^1(\mathbb{R})} \leq C|t_1 - t_2|, \tag{5.4}$$

for some constant $C$ independent of $\Delta$, and where $\Gamma := [0, S_{\max}] \times [Q_{\mathrm{L}}, Q_{\mathrm{R}}]$. The second inequality follows from Lemma 5.1, and the Lipschitz continuity of $\Psi$. By standard arguments, see, e.g., [10], it is straightforward to show that the sequence $\{u^{\Delta}\}_{\Delta>0}$ is compact in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$. Let therefore $z = \lim_{\Delta\to 0} z^{\Delta}$ and define $u(x,t) = \Psi^{-1}(\boldsymbol{\gamma}(x), z(x,t))$. Following [16], we can conclude that $u$ is a weak solution of the initial value problem (3.1), and hence Theorem 5.1 follows.

**Lemma 5.1.** *If the CFL condition* (5.1) *is satisfied for one time step, then the computed solutions remain in the interval* $[0,1]$, (5.1) *holds for each succeeding time step, and the scheme* (3.2) *is monotone. In addition, there exists a constant* $C$*, independent of* $\Delta$*, such that*

$$\sum_{j\in\mathbb{Z}} |U_j^{n+1} - U_j^n| S_j \Delta x \leq \sum_{j\in\mathbb{Z}} |U_j^1 - U_j^0| S_j \Delta x \leq C\Delta t. \tag{5.5}$$

*Proof.* Formula (3.2) defines $U_j^{n+1}$ as a function

$$U_j^{n+1} = G_j(U_{j+1}^n, U_j^n, U_{j-1}^n, \boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j).$$

The partial derivatives with respect to the conserved variables are

$$\begin{aligned}
\frac{\partial G_j}{\partial U_{j\pm1}^n} &= \mp\lambda_j f_u^{\mp}(\boldsymbol{\gamma}_{j\pm\frac12}, U_{j\pm1}^n) \geq 0,\\
\frac{\partial G_j}{\partial U_j^n} &= 1 + \lambda_j f_u^-(\boldsymbol{\gamma}_{j+\frac12}, U_j^n) - \lambda_j f_u^+(\boldsymbol{\gamma}_{j-\frac12}, U_j^n).
\end{aligned}$$

Thus $U_j^{n+1}$ is a nondecreasing function of the conserved variables at the lower time level if

$$1 + \lambda_j f_u^-(\boldsymbol{\gamma}_{j+\frac12}, U_j^n) - \lambda_j f_u^+(\boldsymbol{\gamma}_{j-\frac12}, U_j^n) \geq 0,$$

which holds if the CFL condition (5.1) is valid at $t = t_n$. If (5.1) holds for the initial data, i.e. for $n = 0$, then, since $G(\cdot,\cdot,\cdot,\boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j)$ is nondecreasing as a function of each of its first arguments and $U_j^0 \in [0,1]$,

$$\begin{aligned}
0 &\leq G_j(0,0,0,\boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j)\\
&\leq G_j(U_{j+1}^0, U_j^0, U_{j-1}^0, \boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j)\\
&= U_j^{n+1} \leq G_j(1,1,1,\boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j) \leq 1.
\end{aligned}$$

Here, we have used that $Q_{\mathrm{R}} - Q_{\mathrm{L}} \geq 0$ and the inequalities

$$0 \leq G_j(0,0,0,\boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j) \leq Q_{\mathrm{R}} - Q_{\mathrm{L}},$$

$$\begin{aligned}
&1 - \lambda_j(Q_{\mathrm{R}} - Q_{\mathrm{L}})(1 + u_{\mathrm{F}})\\
&\quad \leq G_j(1,1,1,\boldsymbol{\gamma}_{j+\frac12}, \boldsymbol{\gamma}_{j-\frac12}, \lambda_j) \leq 1.
\end{aligned}$$

Proceeding inductively, it is clear that $U_j^n \in [0,1]$ for each $n \geq 0$, and thus (5.1) remains satisfied at each subsequent time level. The scheme is monotone since $U_j^{n+1}$ is a nondecreasing function of $U_{j+1}^n$, $U_j^n$ and $U_{j-1}^n$.

To prove (5.5), we rewrite (3.2) as

$$\begin{aligned}
&U_j^{n+1} - U_j^n\\
&= U_j^n - U_j^{n-1} - \lambda_j \Delta_-\big(f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac12}, U_{j+1}^n, U_j^n)\\
&\quad - f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac12}, U_{j+1}^{n-1}, U_j^{n-1})\big)\\
&= \big(1 - \lambda_j C_{j+\frac12}^{n-\frac12} + \lambda_j B_{j-\frac12}^{n-\frac12}\big)(U_j^n - U_j^{n-1})\\
&\quad - \lambda_j B_{j+\frac12}^{n-\frac12}(U_{j+1}^n - U_{j+1}^{n-1}) + \lambda_j C_{j-\frac12}^{n-\frac12}(U_{j-1}^n - U_{j-1}^{n-1}),\\
&B_{j+\frac12}^{n-\frac12} := \int_0^1 f_u^-\big(\boldsymbol{\gamma}_{j+\frac12}, \theta U_{j+1}^n + (1-\theta)U_{j+1}^{n-1}\big)\, d\theta \leq 0,\\
&C_{j+\frac12}^{n-\frac12} := \int_0^1 f_u^+\big(\boldsymbol{\gamma}_{j+\frac12}, \theta U_j^n + (1-\theta)U_j^{n-1}\big)\, d\theta \geq 0.
\end{aligned}$$

In view of the CFL condition (5.1), we get

$$1 - \lambda_j C_{j+\frac12}^{n-\frac12} + \lambda_j B_{j-\frac12}^{n-\frac12} \geq 0,$$

and therefore

$$\begin{aligned}
|U_j^{n+1} - U_j^n| &\leq \Big(1 - \lambda_j\big(C_{j+\frac12}^{n-\frac12} - B_{j-\frac12}^{n-\frac12}\big)\Big)|U_j^n - U_j^{n-1}|\\
&\quad - \lambda_j B_{j+\frac12}^{n-\frac12}|U_{j+1}^n - U_{j+1}^{n-1}| + \lambda_j C_{j-\frac12}^{n-\frac12}|U_{j-1}^n - U_{j-1}^{n-1}|.
\end{aligned}$$

Multiplying this inequality by $S_j$, summing it over $j$ and multiplying the result by $\Delta x$ gives

$$\sum_{j\in\mathbb{Z}} |U_j^{n+1} - U_j^n| S_j \Delta x \leq \sum_{j\in\mathbb{Z}} |U_j^n - U_j^{n-1}| S_j \Delta x.$$

Continuing this way by induction yields

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} |U_j^1 - U_j^0| S_j \Delta x = \Delta t \sum_{j\in\mathbb{Z}} \big|\Delta_- f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac12}, U_{j+1}^0, U_j^0)\big|\\
&\leq \Delta t\Bigg[\sum_{j\in\mathbb{Z}} \big|f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac12}, U_{j+1}^0, U_j^0) - f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac12}, U_j^0, U_j^0)\big|\\
&\qquad + \sum_{j\in\mathbb{Z}} \big|f(\boldsymbol{\gamma}_{j+\frac12}, U_j^0) - f(\boldsymbol{\gamma}_{j-\frac12}, U_j^0)\big|\\
&\qquad + \sum_{j\in\mathbb{Z}} \big|f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j-\frac12}, U_j^0, U_j^0) - f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j-\frac12}, U_j^0, U_{j-1}^0)\big|\Bigg]\\
&\leq \Delta t\big(2\|f_u\|\|u_0\|_{BV} + \|\boldsymbol{\gamma}\|_{BV}\big),
\end{aligned}$$

which implies (5.5). ■

In what follows, we will use the Kružkov [11] entropy-entropy flux pair $(V, F)$ indexed by $c$, i.e. $V(u) := |u - c|$ and $F(\boldsymbol{\gamma}, u) := \mathrm{sgn}(u-c)(f(\boldsymbol{\gamma}, u) - f(\boldsymbol{\gamma}, c))$, $c \in \mathbb{R}$, where $\mathrm{sgn}(w) = w/|w|$ if $w \neq 0$ and $\mathrm{sgn}(0) = 0$. We denote by $\mathcal{O}(\Delta\boldsymbol{\gamma}_j)$ terms which sum (over $j$) to $\mathcal{O}(\|\boldsymbol{\gamma}\|_{BV})$. Furthermore, we use the notation $\Delta_+^u$ and $\Delta_-^u$ for spatial difference operators with respect to $u$ only, keeping $\boldsymbol{\gamma}$ fixed. Similarly, $\Delta_+^{\boldsymbol{\gamma}}$ and $\Delta_-^{\boldsymbol{\gamma}}$ denote spatial difference operators with respect to $\boldsymbol{\gamma}$ only with $u$ kept fixed.

**Lemma 5.2.** *For each $c \in \mathbb{R}$, the entropy inequality*

$$\begin{aligned} V(U_j^{n+1}) \leq & V(U_j^n) - \lambda_j \Delta_-^u F^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \\ & + \lambda_j \mathcal{O}(\Delta \boldsymbol{\gamma}_j), \end{aligned} \tag{5.6}$$

*is valid, where the Engquist-Osher numerical entropy flux is defined by*

$$\begin{aligned} F^{\mathrm{EO}}(\boldsymbol{\gamma}, v, u) &:= \frac{1}{2}\big(F(\boldsymbol{\gamma}, u) + F(\boldsymbol{\gamma}, v)\big) \\ & - \frac{1}{2}\int_u^v \operatorname{sgn}(w-c)\big|f_u(\boldsymbol{\gamma}, w)\big|\, dw. \end{aligned} \tag{5.7}$$

*Proof.* Let $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. With

$$\rho_j^{n+1} := U_j^n - \lambda_j \Delta_-^u f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n),$$

the discrete entropy inequality

$$V(\rho_j^{n+1}) \leq V(U_j^n) - \lambda_j \Delta_-^u F^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)$$

is valid, since $F^{\mathrm{EO}}(\boldsymbol{\gamma}, v, u)$ can be written as

$$\begin{aligned} F^{\mathrm{EO}}(\boldsymbol{\gamma}, v, u) = & f^{\mathrm{EO}}(\boldsymbol{\gamma}, v \vee c, u \vee c) \\ & - f^{\mathrm{EO}}(\boldsymbol{\gamma}, v \wedge c, u \wedge c). \end{aligned} \tag{5.8}$$

Then

$$\begin{aligned} V(U_j^{n+1}) \leq & V(U_j^n) - \lambda_j \Delta_-^u F^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \\ & - V(\rho_j^{n+1}) + V(U_j^{n+1}). \end{aligned}$$

It remains to prove $V(\rho_j^{n+1}) - V(U_j^{n+1}) = \lambda_j \mathcal{O}(\Delta \boldsymbol{\gamma}_j)$:

$$\begin{aligned} & \big|V(\rho_j^{n+1}) - V(U_j^{n+1})\big| \leq |\rho_j^{n+1} - U_j^{n+1}| \\ & = \lambda_j \big|\Delta_- f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \\ & \qquad - \Delta_-^u f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)\big| \\ & \leq \lambda_j \big(2\|f_{\boldsymbol{\gamma}}\| + L_u \boldsymbol{\gamma}\big)|\boldsymbol{\gamma}_{j+\frac{1}{2}} - \boldsymbol{\gamma}_{j-\frac{1}{2}}| = \lambda_j \mathcal{O}(\Delta \boldsymbol{\gamma}_j), \end{aligned}$$

where $L_u \boldsymbol{\gamma}$ denotes the Lipschitz constant of $f_u$ with respect to $\boldsymbol{\gamma}$. ∎

We now let $\chi^+(w;c)$ denote the characteristic function for the interval $[c, +\infty)$, $\chi^-(w;c)$ the characteristic function for $(-\infty, c]$. We will also use the notation $a_+ = \max\{a, 0\}$, $a_- = \min\{a, 0\}$.

**Lemma 5.3.** *For any constant $c \in \mathbb{R}$ and for all $j \in \mathbb{Z}$ the following inequalities hold:*

$$\begin{aligned} & \pm \left( \int_{U_j^n}^{U_{j+1}^n} \chi^{\pm}(u;c) f_u^-(\boldsymbol{\gamma}_{j+\frac{1}{2}}, u)\, du \right. \\ & \qquad \left. + \int_{U_{j-1}^n}^{U_j^n} \chi^{\pm}(u;c) f_u^+(\boldsymbol{\gamma}_{j+\frac{1}{2}}, u)\, du \right) \\ & \leq \frac{\pm 1}{\lambda_j}\left(U_j^n - U_j^{n-1}\right)_{\pm} + \mathcal{O}(\Delta \boldsymbol{\gamma}_j). \end{aligned} \tag{5.9}$$

*Proof.* The following two identities are proved in [16]:

$$\begin{aligned} & \frac{1}{2}\big[\Delta_-^u F^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \pm \Delta_-^u f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)\big] \\ & = \pm \int_{U_j^n}^{U_{j+1}^n} \chi_{\pm}(u;c) f_u^-(\boldsymbol{\gamma}_{j+\frac{1}{2}}, u)\, du \\ & \quad \pm \int_{U_{j-1}}^{U_j^n} \chi_{\pm}(u;c) f_u^+(\boldsymbol{\gamma}_{j+\frac{1}{2}}, u)\, du. \end{aligned} \tag{5.10}$$

Using Lemma 5.2, we find that

$$\begin{aligned} & \Delta_-^u F^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \leq \frac{1}{\lambda_j}(V_j^n - V_j^{n+1}) + \mathcal{O}(\Delta \boldsymbol{\gamma}_j) \\ & \leq \frac{1}{\lambda_j}|U_j^n - U_j^{n+1}| + \mathcal{O}(\Delta \boldsymbol{\gamma}_j). \end{aligned} \tag{5.11}$$

Since $f^{\mathrm{EO}}$ is Lipschitz continuous in $\boldsymbol{\gamma}$, we have

$$\begin{aligned} & \Delta_-^u f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \\ & = \Delta_- f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mathcal{O}(\Delta \boldsymbol{\gamma}_j). \end{aligned}$$

The marching formula (3.2) then implies that

$$\Delta_-^u f^{\mathrm{EO}}(\boldsymbol{\gamma}_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) = \frac{1}{\lambda_j}(U_j^n - U_j^{n+1}) + \mathcal{O}(\Delta \boldsymbol{\gamma}_j). \tag{5.12}$$

Adding (5.12) to (5.11), dividing by two, and using (5.10) with "+" we obtain (5.9) with "+". Similarly, subtracting (5.12) from (5.11), dividing by two and using (5.10) with "−" we obtain (5.9) with "−". ∎

The singular mapping $\Psi$ defined in (5.2) will be used to transform the numerical approximations $u^{\Delta}$ into a sequence $z^{\Delta}$ for which compactness can be proved. To this end we state some elementary properties of $\Psi$.

**Lemma 5.4.** *For almost all $x \in \mathbb{R}$, $\Psi(\boldsymbol{\gamma}(x), u)$ is a strictly increasing mapping as a function of $u$, and $\Psi(\boldsymbol{\gamma}, u)$ is Lipschitz continuous with respect to $u$ and $\boldsymbol{\gamma}$.*

*Proof.* From the definition of $f$ we see that

$$f_u\big(\boldsymbol{\gamma}(x), u\big) = \gamma_2(x) + \gamma_1(x) h'(u).$$

For $|x| > 1$, $f_u(\boldsymbol{\gamma}(x), u) = \gamma_2(x)$, which is nonzero because $Q_{\mathrm{L}} < 0$ and $Q_{\mathrm{R}} > 0$. For $x \in (0,1)$, any zeros of $f_u(\boldsymbol{\gamma}(x), u)$ are isolated (with respect to $u$), since $\gamma_1(x) > 0$ and $h$ is genuinely nonlinear. This is also true for $x \in (-1, 0)$. That $\Psi(\boldsymbol{\gamma}(x), u)$ is strictly increasing is now immediate. The easily verified relationships

$$\begin{aligned} \big|\Psi(\boldsymbol{\gamma}, \tilde{u}) - \Psi(\boldsymbol{\gamma}, u)\big| &\leq \|f_u\||\tilde{u} - u|, \\ \big|\Psi(\tilde{\gamma}_1, \gamma_2, u) - \Psi(\gamma_1, \gamma_2, u)\big| &\leq \|h'\||\tilde{\gamma}_1 - \gamma_1|, \\ \big|\Psi(\gamma_1, \tilde{\gamma}_2, u) - \Psi(\gamma_1, \gamma_2, u)\big| &\leq |\tilde{\gamma}_2 - \gamma_2| \end{aligned}$$

imply the Lipschitz continuity assertions. ∎

Our goal is now to construct a certain decomposition of $\Psi$. Let $N_\epsilon(\mathbf{0})$ denote an open disk of radius $\epsilon$ centered at the origin in the $\gamma_1$-$\gamma_2$ plane, and define the set

$$\Gamma_\epsilon := \big([0, S_{\max}] \times [Q_{\mathrm{L}}, Q_{\mathrm{R}}]\big) \backslash N_\epsilon(\mathbf{0}).$$

**Lemma 5.5.** *There are finitely many functions* $u_k^*(\boldsymbol{\gamma})$ *such that*

$$0 = u_0^*(\boldsymbol{\gamma}) \le u_1^*(\boldsymbol{\gamma}) \le \cdots \le u_{m-1}^* \le u_m^*(\boldsymbol{\gamma}) = 1,$$

*and* $|f_u(\cdot,\boldsymbol{\gamma})| > 0$ *on each interval* $(u_k^*(\boldsymbol{\gamma}), u_{k+1}^*(\boldsymbol{\gamma}))$. *In addition, each* $u_k^*$ *depends Lipschitz continuously on the parameter vector* $\boldsymbol{\gamma}$ *in the sense that for any* $\epsilon > 0$ *there is a Lipschitz constant* $M_\epsilon^k$ *such that if the line segment joining* $(\tilde\gamma_1, \tilde\gamma_2)$ *to* $(\gamma_1, \gamma_2)$ *lies entirely within the set* $\Gamma_\epsilon$ *(we refer to this as Condition L), then*

$$\left|u_k^*(\tilde\gamma_1,\tilde\gamma_2) - u_k^*(\gamma_1,\gamma_2)\right| \le M_\epsilon^k\left(|\tilde\gamma_1 - \gamma_1| + |\tilde\gamma_2 - \gamma_2|\right). \tag{5.13}$$

*Finally, for the discretized version of* $\boldsymbol{\gamma}$, *each adjacent pair of values* $\boldsymbol{\gamma}_{j-\frac12}$ *and* $\boldsymbol{\gamma}_{j+\frac12}$ *satisfies Condition L with any value of* $\epsilon$ *for which* $\epsilon < \min\left(-Q_{\mathrm{L}}, Q_{\mathrm{R}}, S_{\min}\right)$.

*Proof.* We start by observing that for fixed $\boldsymbol{\gamma}$ the critical points of $f(\boldsymbol{\gamma},\cdot)$ satisfy the equation $\gamma_2 + \gamma_1 h'(u) = 0$. We ignore for the moment the possibility that $\gamma_1 = 0$, which allows us to rewrite this as $-h'(u) = \gamma_2/\gamma_1$. This makes it clear that for $\gamma_1 \ne 0$, the critical points of $f(\boldsymbol{\gamma}, u)$ (the solutions of $f_u(\boldsymbol{\gamma}, u) = 0$) are located in the $u$-$z$ plane at the intersection of the graph of $z = -h'(u)$ with the horizontal line $z = \gamma_2/\gamma_1$. For any single horizontal line of this type, there are only finitely many such intersections; this is a consequence of genuine nonlinearity.

In the $u$-$z$ plane, we now construct a finite number of curves $\mathcal{C}_0, \ldots, \mathcal{C}_m$ as follows (see Figure 5.1): The curve $\mathcal{C}_0$ is just the vertical line $u = 0$. The definition of $\mathcal{C}_1$ starts with the portion of $z = -h'(u)$ between $u = 0$ and the first turning point of $z = -h'$. If the first turning point is a maximum (minimum), we continue the curve along the vertical half line lying above (below) the graph of $z = -h'$. We also continue the curve $\mathcal{C}_1$ along the vertical line $u = 0$, choosing the upper half line or lower half line in such a way that $z$ is always increasing or decreasing along $\mathcal{C}_1$. We then define $\mathcal{C}_2$ in a similar way, connecting the first and second critical points of $z = -h'$, and then continuing the curve along vertical half lines consistent with the monotonicity of $z = -h'$. Continuing this way, the final curve $\mathcal{C}_m$ will be the vertical line $u = 1$. By construction, a horizontal line of height $z$ intersects each curve $\mathcal{C}_k$ exactly once. Let $v_k^*(z)$ denote the (unique) intersection at the curve $\mathcal{C}_k$. It is clear that each $v_k^*(z)$ is a Lipschitz continuous function of $z$, that $v_k^* \le v_{k+1}^*$, and that each $v_k^*$ is constant for $|z|$ sufficiently large. More specifically, there are constants $\underline{z}_k \le \overline{z}_k$, and $v_k^*(-\infty)$, $v_k^*(+\infty)$ such that $v_k^*(z) = v_k^*(-\infty)$ for $z \le \underline{z}_k$ and $v_k^*(z) = v_k^*(+\infty)$ for $z \ge \overline{z}_k$.

Now recalling that $z = \gamma_2/\gamma_1$, we define

$$u_k^*(\boldsymbol{\gamma}) = \begin{cases} v_k^*(-\infty) & \text{for } \gamma_2 \le \underline{z}_k\gamma_1, \\ v_k^*(\gamma_2/\gamma_1) & \text{for } \underline{z}_k\gamma_1 \le \gamma_2 \le \overline{z}_k\gamma_1, \\ v_k^*(+\infty) & \text{for } \gamma_2 \ge \overline{z}_k\gamma_1, \end{cases} \tag{5.14}$$

and observe that $u_k^*$ is well-defined in the closed right half-plane $\{\gamma | \gamma_1 \ge 0\}$, except possibly at the origin (due to differing radial limits). Moreover, if we exclude some neighborhood $N_\epsilon(\mathbf{0})$ of the origin, $u_k^*(\boldsymbol{\gamma})$ inherits Lipschitz continuity from $v_k(z)$. Indeed, given $\gamma$ and $\tilde\gamma$ such that the line segment connecting them lies in $\Gamma_\epsilon$, if we integrate the gradient of $u_k^*$ along that line segment, we get the Lipschitz estimate (5.13).

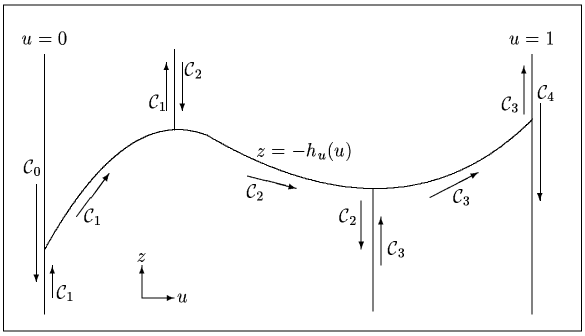


**Fig. 5.1.** The curves $\mathcal{C}_k$ defined in the proof of Lemma 5.5. Any horizontal line $z =$ const. intersects each $\mathcal{C}_k$ exactly once.

For each (relevant) value of $\boldsymbol{\gamma}$, define the following intervals, whose endpoints depend on $\boldsymbol{\gamma}$:

$$\mathcal{I}_k(\boldsymbol{\gamma}) := \left[u_k^*(\boldsymbol{\gamma}), u_{k+1}^*(\boldsymbol{\gamma})\right), \quad k = 0, \ldots, m-1.$$

By construction, the union of these intervals equals $[0,1]$, and each of these intervals has Lipschitz continous length as long as the line segment joining the endpoints lies in $\Gamma_\epsilon$. It is clear that by construction, there are no zeros of $f_u$ in the interior of $\mathcal{I}_k$ (which may be empty), and so $f_u$ is strictly monotone in the interior of $\mathcal{I}_k$.

We only have to check that $\boldsymbol{\gamma}_{j-\frac12}$ and $\boldsymbol{\gamma}_{j+\frac12}$ satisfy Condition L. Note first that we always have (with the notation $\boldsymbol{\gamma}_{j+\frac12} =: (\gamma_{j+\frac12,1}, \gamma_{j+\frac12,2})$) $Q_{\mathrm{L}} \le \gamma_{j-\frac12,2} \le Q_{\mathrm{R}}$ and $Q_{\mathrm{L}} \le \gamma_{j+\frac12,2} \le Q_{\mathrm{R}}$. If the intervals $I_{j-\frac12}$ and $I_{j+\frac12}$ lie entirely within $(-1,1)$, then $S_{\min} \le \gamma_{j-\frac12,1} \le S_{\max}$ and $S_{\min} \le \gamma_{j+\frac12,1} \le S_{\max}$. Therefore, if $\epsilon < S_{\min}$, Condition L is satisfied. If the intervals $I_{j-\frac12}$ and $I_{j+\frac12}$ lie entirely within $(-\infty, 0)$, then Condition L is satisfied if $\epsilon < -Q_{\mathrm{L}}$. Similarly, if the intervals $I_{j-\frac12}$ and $I_{j+\frac12}$ lie entirely within $(0,\infty)$, then Condition L is satisfied as long as $\epsilon < Q_{\mathrm{R}}$. Since any pair of intervals $I_{j-\frac12}$, $I_{j+\frac12}$ satisfies one of these three conditions, the proof is complete. ∎

Letting $\chi(w; \mathcal{I}(\boldsymbol{\gamma}))$ denote the characteristic function of $\mathcal{I}(\boldsymbol{\gamma})$, we can now write down the decomposition of $\Psi$:

$$\begin{aligned} \Psi(\boldsymbol{\gamma}, u) &= \sum_{k=0}^{m} \int_0^u \chi(w; \mathcal{I}_k(\boldsymbol{\gamma}))\left|f_u(\boldsymbol{\gamma}, w)\right| dw \\ &=: \sum_{k=0}^{m} \Psi^k(\boldsymbol{\gamma}, u). \end{aligned}$$

Since $f_u$ is Lipschitz continuous in $\gamma$, and the endpoints of each $\mathcal{I}_k(\tilde\gamma)$ are Lipschitz continuous if Condition L is satisfied, we have

$$\begin{aligned} &\left|\Psi^k(\tilde\gamma, u) - \Psi^k(\gamma, u)\right| \\ &\le \int_0^u \Big\{ \Big|\chi\big(w; \mathcal{I}_k(\tilde\gamma)\big) - \chi\big(w; \mathcal{I}_k(\gamma)\big)\Big| \left|f_u(\tilde\gamma, w)\right| \end{aligned}$$

$$+\chi\big(w;\mathcal{I}_k(\gamma)\big)\Big|\big|f_u(\gamma,w)\big|-\big|f_u(\tilde{\gamma},w)\big|\Big|\Big\}dw$$
$$\leq M|\tilde{\gamma}-\gamma|$$

for some constant $M$ depending only on $f$, and hence $\Psi^k$ is Lipschitz continuous in $\gamma$ as long as Condition L is satisfied.

For a fixed $i \in \{0,\dots,m\}$, let $P_+^i$ ($N_+^i$) denote those integers $k$, $i \leq k \leq m$, such that $f_u \geq 0$ ($\leq 0$) in the interval $\mathcal{I}_k$. Similarly, we let $P_-^i$ ($N_-^i$) denote those integers $k$, $k < i \leq m$, such that $f_u > 0$ ($< 0$) on $\mathcal{I}_k$. With these definitions, we can state the following lemma:

**Lemma 5.6.** *For each $i \in \{0,\dots,m\}$ the following pair of inequalities holds:*

$$-\sum_{k\in N_+^i} \Delta_+^u \Psi^k\left(\gamma_{j+\frac{1}{2}},U_j^n\right)+\sum_{k\in P_+^i} \Delta_-^u \Psi^k\left(\gamma_{j-\frac{1}{2}},U_j^n\right) \leq \frac{1}{\lambda_j}\left(U_j^n-U_j^{n+1}\right)_+ + \mathcal{O}(\Delta\gamma_j), \tag{5.15}$$

$$\sum_{k\in N_-^i} \Delta_+^u \Psi^k\left(\gamma_{j+\frac{1}{2}},U_{j+1}^n\right)-\sum_{k\in P_-^i} \Delta_-^u \Psi^k\left(\gamma_{j-\frac{1}{2}},U_j^n\right) \leq -\frac{1}{\lambda_j}\left(U_j^n-U_j^{n+1}\right)_- + \mathcal{O}(\Delta\gamma_j). \tag{5.16}$$

*Proof.* Recalling $\chi^+(u;u_i^*(\gamma_{j+\frac{1}{2}}))=0$ for $u<u_i^*(\gamma_{j+\frac{1}{2}})$, we find that

$$\begin{aligned}
&\int_{U_j^n}^{U_{j+1}^n} \chi^+\big(u;u_i^*(\gamma_{j+\frac{1}{2}})\big)f_u^-\left(\gamma_{j+\frac{1}{2}},u\right)\,du\\
&=\sum_{k\in N_+^i}\int_{U_j^n}^{U_{j+1}^n}\chi\big(w;\mathcal{I}_k(\gamma_{j+\frac{1}{2}})\big)f_u^-\left(\gamma_{j+\frac{1}{2}},u\right)\,du \qquad (5.17)\\
&=-\sum_{k\in N_+^i}\Delta_-^u\Psi^k\left(\gamma_{j+\frac{1}{2}},U_{j+1}^n\right).
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&\int_{U_{j-1}^n}^{U_j^n} \chi^+\big(u;u_i^*(\gamma_{j+\frac{1}{2}})\big)f_u^+\left(\gamma_{j+\frac{1}{2}},u\right)\,du\\
&=\sum_{k\in P_+^i}\Delta_-^u\Psi^k\left(\gamma_{j+\frac{1}{2}},U_j^n\right),
\end{aligned}\tag{5.18}$$

$$\begin{aligned}
&\int_{U_j^n}^{U_{j+1}^n} \chi^-\big(u;u_i^*(\gamma_{j+\frac{1}{2}})\big)f_u^-\left(\gamma_{j+\frac{1}{2}},u\right)\,du\\
&=-\sum_{k\in N_-^i}\Delta_-^u\Psi^k\left(\gamma_{j+\frac{1}{2}},U_{j+1}^n\right),
\end{aligned}\tag{5.19}$$

$$\begin{aligned}
&\int_{U_{j-1}^n}^{U_j^n} \chi^-\big(u;u_i^*(\gamma_{j+\frac{1}{2}})\big)f_u^+\left(\gamma_{j+\frac{1}{2}},u\right)\,du\\
&=\sum_{k\in P_-^i}\Delta_-^u\Psi^k\left(\gamma_{j+\frac{1}{2}},U_j^n\right).
\end{aligned}\tag{5.20}$$

To conclude the proof we must replace the $\Delta^u$ difference operators by $\Delta$ operators. Recalling that $\Psi^k$ is Lipschitz continuous in $\gamma$, we can replace $\gamma_{j+\frac{1}{2}}$ with $\gamma_{j-\frac{1}{2}}$ where required, absorbing the difference in the $\mathcal{O}(\Delta\gamma_j)$ term. Then we use (5.17) and (5.18) (with $\gamma_{j-\frac{1}{2}}$) in (5.9) (with "+") and obtain (5.15). The inequality (5.16) is similarly obtained from (5.19), (5.20), and (5.9) (with "−"). ∎

**Lemma 5.7.** *For each $k=0,\dots,m$, there exists a constant $C^k \geq 0$, independent of $\Delta$ and $n$, such that*

$$\sum_{j\in\mathbb{Z}}\left|\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right| \leq C^k. \tag{5.21}$$

*Proof.* From the definition of $\Psi^k$ and $U_j^n \in [0,1]$, it follows that $|\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)| \leq \|f_u\|$. Thus, for each $J>0$

$$\left|\sum_{j=-J}^{J}\Delta_+\Psi^k\left(\gamma_{j+\frac{1}{2}},U_j^n\right)\right| \leq 2\|f_u\|, \tag{5.22}$$

$$\begin{aligned}
&\sum_{j=-J}^{J}\left|\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right|\\
&=2\sum_{j=-J}^{J}\left(\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right)_+-\sum_{j=-J}^{J}\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\\
&=2\sum_{j=-J}^{J}\left(\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right)_+-\sum_{j=-J}^{J}\Delta_+\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\\
&\quad+\sum_{j=-J}^{J}\Delta_+^\gamma\Psi^k(\gamma_{j+\frac{1}{2}},U_{j+1}^n) \qquad (5.23)\\
&\leq 2\sum_{j=-J}^{J}\left(\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right)_+ + 2\|f_u\|+\mathcal{O}(\|\gamma\|_{BV}).
\end{aligned}$$

Letting $J\to\infty$, we get

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left|\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right|\\
&\leq 2\sum_{j\in\mathbb{Z}}\left(\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right)_+ + 2\|f_u\|+\mathcal{O}(\|\gamma\|_{BV}).
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left|\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right| \qquad (5.24)\\
&\leq -2\sum_{j\in\mathbb{Z}}\left(\Delta_+^u\Psi^k(\gamma_{j+\frac{1}{2}},U_j^n)\right)_- + 2\|f_u\|+\mathcal{O}(\|\gamma\|_{BV}).
\end{aligned}$$

Without loss of generality, we can assume that $f_u > 0$ for $u \in I_0$. By this assumption $N_-^0 = \varnothing$ and $P_-^0 = \{0\}$, hence (5.16) with $i=0$ reads

$$\begin{aligned}
&-\left(\Psi^0(\gamma_{j-\frac{1}{2}},U_j^n)-\Psi^0(\gamma_{j-\frac{1}{2}},U_{j-1}^n)\right)\\
&\leq -\frac{1}{\lambda_j}\left(U_j^n-U_j^{n+1}\right)_- + \mathcal{O}(\Delta\gamma_j).
\end{aligned}$$

Since the right hand side of this equation is nonnegative, it follows that

$$-\sum_{j\in\mathbb{Z}}\left(\Delta_+^u\Psi^0(\gamma_{j+\frac{1}{2}},U_j^n)\right)_-$$

$$\leq \sum_{j\in\mathbb{Z}} \frac{1}{\lambda_j}|U_j^1 - U_j^0| + \mathcal{O}(\|\gamma\|_{BV}).$$

Now using (5.24) produces the estimate

$$\sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi^0(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr|$$
$$\leq 2\sum_{j\in\mathbb{Z}} \frac{1}{\lambda_j}|U_j^1 - U_j^0| + \mathcal{O}(\|\gamma\|_{BV}) =: C_0.$$

Now we set $i = 2$ in (5.16), and recall that $P_-^1 = \{0\}$ and $N_-^1 = \{1\}$, in this case we find

$$\Delta_+^u \Psi^1(\gamma_{j+\frac{1}{2}}, U_j^n) + \Delta_+^u \Psi^0(\gamma_{j-\frac{1}{2}}, U_{j-1}^n)$$
$$\leq -\frac{1}{\lambda_j}\left(U_j^n - U_{j-1}^n\right)_- + \mathcal{O}(\Delta\gamma_j).$$

Rearranging this, we see that

$$\sum_{j\in\mathbb{Z}} \bigl(\Delta_+^u \Psi^1(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr)_+$$
$$\leq \sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi^0(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr| - \frac{1}{\lambda_j}\left(U_j^n - U_{j-1}^n\right)_-$$
$$+ \mathcal{O}(\Delta\gamma_j),$$

and thus by (5.23)

$$\sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi^1(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr| \leq 2C^0 + C^0 =: C^1.$$

Continuing inductively for $i = 2, 3, \dots, m-1$, we get

$$\sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi^i(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr| \leq C^0 + 2\sum_{k=0}^{i-1} C^k =: C^i.$$

For $i = m$, we use (5.15) and proceed as in the case where $i = 0$ to find that

$$\sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi^m(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr| \leq C^0 =: C^m,$$

which concludes the proof. ∎

Consequently, we have that

$$\|z^\Delta(\cdot, t)\|_{BV}$$
$$= \sum_{j\in\mathbb{Z}} \bigl|\Delta_+^u \Psi(\gamma_{j+\frac{1}{2}}, U_j^n)\bigr| + \bigl|\Delta_+^\gamma \Psi(\gamma_{j-\frac{1}{2}}, U_j^n)\bigr|$$
$$\leq \sum_{k=0}^{m} C^k + \mathcal{O}\bigl(\|\gamma\|_{BV}\bigr),$$

for all $t \geq 0$. Using this bound together with the obvious bounds, we obtain the desired estimates (5.3) and (5.4).

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