A conservation law with discontinuous flux modelling traffic flow with abruptly changing road surface conditions

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Abstract. We consider a scalar conservation law with a flux that depends discontinously on the space variable, with a focus on a simple kinematic traffic model in which both the maximum speed and number of lanes vary abruptly. Such a conservation law admits many L^1 contraction semigroups, one for each so-called connection (A, B). Here we define entropy solutions of type (A, B) involving Kružkov-type entropy inequalities that can be adapted to any fixed connection (A, B), and which imply the L^1 contraction property for L^∞ solutions. We outline the proof of convergence of a new difference scheme that approximates entropy solutions of type (A, B) for any connection (A, B) if a few parameters are varied. The scheme relies on a modification of the standard Engquist-Osher flux. Some numerical examples are presented.

1. Introduction

We are interested in the numerical approximation of solutions to the initial value problem for a scalar conservation law

$$u_t + \mathcal{F}(x, u)_x = 0, \quad (x, t) \in \Pi_T := \mathbb{R} \times (0, T); \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}; \tag{1.1}$$

$$\mathcal{F}(x, u) := H(x)f(u) + (1 - H(x))g(u), \tag{1.2}$$

where $H(x)$ is the Heaviside function, such that the spatial dependence of the flux $\mathcal{F}(x, u)$ is discontinuous at $x = 0$ if the functions f and g are different. Because of their interesting features and numerous applications, conservation laws with discontinuous flux have seen great interest in recent years, see [**7**] for references.

In a series of papers [**9, 10, 11, 13, 14**], we developed a well-posedness theory for a class of equations with discontinuous flux and constructed simple scalar difference schemes that provably converge to entropy solutions as the discretization parameters tend to zero. These results were extended and applied to some applicative models, see e.g. [**6**]. In the aforementioned papers, we proved the uniqueness

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of weak solutions satisfying a particular entropy condition, provided that either the graphs of g and f do not cross, or the graph of g lies above that of f to the left of any crossing point. In this work, we provide a Kružkov-type entropy formulation that does *not* require this "crossing condition" to ensure uniqueness.

Adimurthi et al. [**2**] pointed out that (1.1) admits many L^1-contractive semi-groups of solutions, one for each so-called *connection* (A, B). A connection is a particular pair of solution values satisfying the Rankine-Hugoniot (RH) condition valid across $x = 0$, i.e., $g(A) = f(B)$. This pair can be adapted to describe a particular physical reality. We put forward a notion of *entropy solutions of type* (A, B) involving distinct Kružkov-type entropy inequalities that can be adapted to any fixed connection (A, B) [**1, 2, 6**]. We prove that these entropy inequalities imply the L^1 contraction property. For a fixed connection, they include a single adapted entropy of the type presented in [**3**]. Our approach avoids the restrictive "piecewise smoothness" setting of [**2**]. We emphasize that an entropy solution "of type (A, B)" is, of course, in general a transient one, which is just "labeled" by (A, B) since these parameters play a role in the solution concept.

This paper summarizes results presented in [**7**] combined with those of [**5**]. Its main contribution is a scalar monotone difference scheme, for which we outline the proof of convergence to entropy solutions of type (A, B) of (1.1). The scheme takes the form of an explicit conservative marching formula on a rectangular grid, where the numerical flux is the Engquist-Osher (EO) flux, with the exception of the cell interface that is associated with the flux discontinuity, and for which a specific interface flux, which is based on a novel modification of the EO flux, is used.

In this paper, we consider the equation (1.1) in the context of the Lighthill-Whitham-Richards traffic model, where we assume that

$$f(u) = v_{\mathrm{R}} u V(u/a_{\mathrm{R}}), \quad g(u) = v_{\mathrm{L}} u V(u/a_{\mathrm{L}}), \tag{1.3}$$

u is the density of cars on a one-directional highway, which is assumed to exhibit both a change of maximum velocity and of the number of lanes at $x = 0$; in fact, v_{L} and v_{R} are the maximum velocities and a_{L} and a_{R} are the numbers of lanes for $x < 0$ and $x > 0$, respectively. The hindrance function $z \mapsto V(z)$ is here assumed to equal $V(z) = (1 - z)^2$ for $0 \le z \le 1$ and $V(z) = 0$ otherwise.

The case with $v_{\mathrm{L}} = v_{\mathrm{R}}$, but $a_{\mathrm{L}} \neq a_{\mathrm{R}}$ has been studied extensively in [**5**]. Our prime motivation to outline our general result in the present context of the traffic model lies in the fact that varying v_{L}, v_{R}, a_{L} and a_{R} we may produce examples of flux crossings that do not satisfy the "crossing condition".

2. Preliminaries

We now state the assumptions on (1.1). Let $a_{\mathrm{L}}, a_{\mathrm{R}} > 0$. We assume that $g \in \mathrm{Lip}([0, a_{\mathrm{L}}])$, $f \in \mathrm{Lip}([0, a_{\mathrm{R}}])$, $f(0) = f(a_{\mathrm{R}}) = g(0) = g(a_{\mathrm{L}}) = 0$, f has a single maximum at $u_f^* \in [0, a_{\mathrm{R}}]$, g has a single maximum at $u_g^* \in [0, a_{\mathrm{L}}]$, f is strictly increasing on $(0, u_f^*)$ and strictly decreasing on (u_f^*, a_{R}), and g is strictly increasing on $(0, u_g^*)$ and strictly decreasing on (u_g^*, a_{L}). We assume that

$$f \text{ and } g \text{ are not linear on any non-degenerate interval} \tag{2.1}$$

and that there is at most one intersection point $u_\chi \in (0, \min(a_{\mathrm{L}}, a_{\mathrm{R}}))$ where $f(u_\chi) = g(u_\chi)$, and if there is such a point u_χ, then the graphs of f and g are assumed to intersect transversally at u_χ (a flux crossing). (Figure 1 shows some possible

configurations of f and g.) For the initial data, we assume that

$$u_0 \in L^\infty(\mathbb{R}); \quad u_0(x) \in [0, a_\mathrm{L}] \text{ for a.e. } x < 0, \quad u_0(x) \in [0, a_\mathrm{R}] \text{ for a.e. } x > 0. \tag{2.2}$$

Fixing $t \in \mathcal{T} := (0,T)$, we define $u_\pm := u(0^\pm, t)$. As we point out below, these traces always exist due to (2.1). Any weak solution of (1.1) will satisfy

$$g(u_-) = f(u_+) \quad \text{(RH condition)}. \tag{2.3}$$

It is well known that (2.3) is not sufficient to guarantee uniqueness, and so additional conditions for a jump across $x = 0$ to be admissible are required. In [**10, 11**] we required, in addition, the so-called *weak characteristic condition* to hold, which states for a pair (u_-, u_+) satisfying (2.3) that

$$\min\{0, g'(u_-)\} \max\{0, f'(u_+)\} = 0 \quad \text{if } u_- \neq u_+. \tag{2.4}$$

This condition requires that the characteristics lead backward toward the x-axis on at least one side of the jump, unless $u_- = u_+$. If $u_- = u_+$, there is no restriction on the characteristics. Adimurthi et al. [**1**] imposed the more restrictive *strong characteristic condition* for a pair (u_-, u_+) satisfying (2.3), which demands that

$$\min\{0, g'(u_-)\} \max\{0, f'(u_+)\} = 0, \tag{2.5}$$

i.e., the characteristics lead backward toward the x-axis on at least one side of the jump, including the case $u_- = u_+$. Both entropy jump conditions, (2.4) and (2.5), result in an L^1-contractive semigroup of solutions. There are other such L^1-contractive semigroups, each associated with a different entropy jump condition (see [**2, 8**]) and characterized by a so-called *connection* (A, B), where in the present context a pair of states (A, B) is called a *connection* if $g(A) = f(B)$, $u_g^* \leq A \leq a_\mathrm{L}$ and $0 \leq B \leq u_f^*$. Figure 1 shows some examples of (A, B)-connections. The entropy jump condition associated with such a connection may be stated as follows.

Definition 2.1. If (u_-, u_+) satisfies (2.3) and (A, B) is a connection, then (u_-, u_+) is said to satisfy the (A, B)-*characteristic condition* if

$$\min\{0, g'(u_-)\} \max\{0, f'(u_+)\} = 0 \quad \text{if } (u_-, u_+) \neq (A, B). \tag{2.6}$$

In Fig, 1, the plots associated with Examples 1, 2, and 3 show flux crossings. Of those, Examples 2 and 3 satisfy the so-called *crossing condition*, which for our fluxes f and g can be stated as follows: for all $u, v \in [0, \min\{a_\mathrm{L}, a_\mathrm{R}\}]$ such that $f(u) - g(u) < 0 < f(v) - g(v)$, we have that $u < v$. In Example 1 shown in Fig, 1, the flux crossing violates the crossing condition, and thus this flux configuration is not covered by the entropy solution theory of [**11**]. We conjectured in [**11**] that our inability to derive a uniqueness result for this configuration was most likely due to some missing entropy condition. It now appears that this entropy condition is the one that derives from the adapted entropy concept that we discuss below.

To state our concept of entropy solution, we need the following function, which is associated with a fixed connection (A, B), and jumps from A to B at $x = 0$:

$$c^{AB}(x) := H(x)B + (1 - H(x))A. \tag{2.7}$$

The function $u \mapsto |u - c^{AB}(x)|$ is an example of an *adapted entropy* [**3**]. Now, we say that a function u is an *entropy solution of type* (A, B) of (1.1) provided it is a weak solution, it satisfies the Kružkov entropy condition to the left and right of $x = 0$, and in addition on Π_T it satisfies the following Kružkov-type entropy inequality:

$$|u - c^{AB}(x)|_t + (\operatorname{sgn}(u - c^{AB}(x))(\mathcal{F}(x, u) - \mathcal{F}(x, c^{AB}(x))))_x \leq 0 \quad \text{in } \mathcal{D}'. \tag{2.8}$$

Condition (2.8) gives us the (A,B)-characteristic condition (2.6), cf. Lemma 3.3.

To construct approximate entropy solutions of type (A,B), we propose a simple scalar upwind difference scheme based on a modification of the EO flux. By modifying a few parameters of the scheme, we can capture the solution associated with any connection (A,B). The scheme is based on the standard EO flux

$$h^{\mathrm{EO}}(u_{\mathrm{R}},u_{\mathrm{L}}) = \frac{1}{2}(q(u_{\mathrm{R}})+q(u_{\mathrm{L}})) - \frac{1}{2}\int_{u_{\mathrm{L}}}^{u_{\mathrm{R}}} |q'(w)|\,dw \tag{2.9}$$

(for $u_t + q(u)_x = 0$), and then, to handle the jump in the flux from g to f, we replace (2.9) by an interface flux h^{AB} (see (4.3) in Sect. 4).

3. Entropy solutions of type (A,B)

Definition 3.1 (Entropy solution of type (A,B)). Let (A,B) be a connection, with c^{AB} defined by (2.7). A measurable function $u : \Pi_T \to R$ is an *entropy solution of type* (A,B) of (1.1) if it satisfies the following conditions:

(**D**.1) $u \in L^\infty(\Pi_T)$; in fact, $u(x,t) \in [0,a_{\mathrm{L}}]$ for a.e. $(x,t) \in \mathbb{R}^- \times \mathcal{T}$ and $u(x,t) \in [0,a_{\mathrm{R}}]$ for a.e. $(x,t) \in \mathbb{R}^+ \times \mathcal{T}$.

(**D**.2) For all test functions $\phi \in \mathcal{D}(\mathbb{R}\times[0,T))$

$$\iint_{\Pi_T} (u\phi_t + \mathcal{F}(x,u)\phi_x)\,dx\,dt + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx = 0. \tag{3.1}$$

(**D**.3) For any test function $0 \le \phi \in \mathcal{D}(\mathbb{R}\times[0,T))$ which vanishes for $x \ge 0$,

$$\iint_{\Pi_T} (|u-c|\phi_t + \operatorname{sgn}(u-c)(g(u)-g(c))\phi_x)\,dx\,dt + \int_{\mathbb{R}} |u_0 - c|\phi(x,0)\,dx \ge 0,$$

and for any test function $0 \le \phi \in \mathcal{D}(\mathbb{R}\times[0,T))$ which vanishes for $x \le 0$,

$$\iint_{\Pi_T} (|u-c|\phi_t + \operatorname{sgn}(u-c)(f(u)-f(c))\phi_x)\,dx\,dt + \int_{\mathbb{R}} |u_0 - c|\phi(x,0)\,dx \ge 0.$$

(**D**.4) The following inequality holds for any test function $0 \le \phi \in \mathcal{D}(\Pi_T)$:

$$\iint_{\Pi_T} \{|u-c^{AB}(x)|\phi_t + \operatorname{sgn}(u-c^{AB}(x))(\mathcal{F}(x,u)-\mathcal{F}(x,c^{AB}(x)))\phi_x\}\,dx\,dt \ge 0. \tag{3.2}$$

A function $u : \Pi_T \to \mathbb{R}$ satisfying conditions (**D**.1) and (**D**.2) is called a weak solution of the initial value problem (1.1).

Equation (3.1) is the weak formulation of the conservation law, and it implies the RH condition (2.3). Conditions (**D**.3) guarantee that the solution is a Kružkov entropy solution of $u_t + g(u)_x = 0$ in $(-\infty,0)\times[0,T)$ and of $u_t + f(u)_x = 0$ in $(0,\infty)\times[0,T)$, respectively. Finally, inequality (3.2) ultimately gives us the (A,B)-characteristic condition (2.6), cf. part **J3** of Lemma 3.3 below.

Lemma 3.2. *Let u be an entropy solution of type (A,B) of (1.1). For a.e. $t \in \mathcal{T}$, the function $u(\cdot,t)$ has strong traces from the left and right at $x = 0$, i.e., the following limits exist for a.e. $t \in \mathcal{T}$: $u(0^-,t) := \operatorname{ess\,lim}_{x\uparrow 0} u(x,t)$, $u(0^+,t) := \operatorname{ess\,lim}_{x\downarrow 0} u(x,t)$. Similarly, u has a strong trace at the initial hyperplane $t = 0$.*

PROOF. The first condition in (**D**.3) guarantees that u is an entropy solution of $u_t + g(u)_x = 0$ on $\mathbb{R}^- \times \mathcal{T}$. This observation, along with assumption (2.1) and a result in [**12**] or [**15**] ensure the existence of a strong trace from the left. The existence of a strong trace from the right follows in a similar way. Finally, the strong trace at the initial hyperplane $t = 0$ follows along the same lines. $\square$

Since the existence of strong traces is guaranteed, we now may describe the behavior of solutions at the interface $x = 0$. The following lemma is proved in [**7**].

LEMMA 3.3. *Let $u_\pm(t) = u(0^\pm, t)$, where u is an entropy solution of type (A, B).*
J1. *The RH condition $f(u_+(t)) = g(u_-(t))$ holds for a.e. $t \in \mathcal{T}$.*
J2. *The following entropy jump condition holds for a.e. $t \in \mathcal{T}$:*

$$\operatorname{sgn}(u_+(t) - B)(f(u_+(t)) - f(B)) - \operatorname{sgn}(u_-(t) - A)(g(u_-(t)) - g(A)) \le 0. \tag{3.3}$$

J3. *For a.e. $t \in \mathcal{T}$, the following characteristic condition is satisfied:*

$$\min\{0, g'(u_-(t))\} \max\{0, f'(u_+(t))\} = 0 \quad \text{if } (u_-(t), u_+(t)) \neq (A, B). \tag{3.4}$$

REMARK 3.4. Condition (3.4) is just the (A, B) characteristic condition (2.6). Reference [**2**] uses (3.3) as the *defining* entropy condition at the interface. We instead *derive* it from the Kružkov-type entropy inequality (3.2). Thus, we do not need the regularity assumption (that the solution is piecewise smooth) required in [**2**], while our solution concept is otherwise equivalent to that of [**2**]. Moreover, discrete versions of the entropy inequalities in Definition 3.1 enable us to prove in a straightforward manner that approximate solutions generated by our numerical scheme converge to entropy solutions of type (A, B).

THEOREM 3.5. *Define $M := \max\{\max_{u \in [0, a_R]} |f'(u)|, \max_{u \in [0, a_L]} |g'(u)|\}$, and let u and v be two entropy solutions of type (A, B) of (1.1), with initial data u_0 and v_0 satisfying (2.2). Then for a.e. $t \in \mathcal{T}$*

$$\int_{-r}^{r} |u(x, t) - v(x, t)|\, dx \le \int_{-r-Mt}^{r+Mt} |u_0(x) - v_0(x)|\, dx \qquad \forall r > 0. \tag{3.5}$$

In particular, there exists at most one entropy solution of type (A, B) of (1.1).

PROOF. Following [**11**], we can establish that for any $0 \le \phi \in \mathcal{D}(\Pi_T)$,

$$-\iint_{\Pi_T} (|u - v|\phi_t + \operatorname{sgn}(u - v)(\mathcal{F}(x, u) - \mathcal{F}(x, v))\phi_x)\, dt\, dx \le E, \tag{3.6}$$

where $E = \int_0^T [\operatorname{sgn}(u - v)(\mathcal{F}(x, u) - \mathcal{F}(x, v))]_{x=0-}^{x=0+} \phi(0, t)\, dt$ and $[\cdot]_{x=0-}^{x=0+}$ indicates the limit from the right minus that from the left at $x = 0$. In [**7**] we prove that $E \le 0$ by demonstrating that the integrand of E is non-negative for almost all $t \in \mathcal{T}$. $\square$

4. Numerical scheme

We define $I_{j+1/2} := [x_j, x_{j+1})$, $x_j = j\Delta x$, $j \in \mathbb{Z}$, and discretize $\mathcal{T}$ via $t^n = n\Delta t$ for $n = 0, \dots, N$, where $N = \lfloor T/\Delta t \rfloor + 1$, and define $I^n := [t^n, t^{n+1})$ for $n = 0, \dots, N - 1$. When sending $\Delta \to 0$, we will keep $\lambda := \frac{\Delta t}{\Delta x}$ constant. Let $\chi_{j+1/2}^n$ be the characteristic function for $I_{j+1/2} \times I^n$. We denote by U_j^n the finite-difference approximation of $u(x_{j+1/2}, t^n)$, and define $u^\Delta(x, t) := \sum_{n \ge 0} \sum_{j \in \mathbb{Z}} U_j^n \chi_{j+1/2}^n(x, t)$. Our difference scheme is an explicit time-marching algorithm of the type

$$U_j^{n+1} = U_j^n - \lambda \Delta_+ \bar{\mathcal{F}}_{j-1/2}^n, \quad U_j^0 := \frac{1}{\Delta x} \int_{I_{j+1/2}} u_0(x)\, dx. \tag{4.1}$$

where we define $\Delta_\pm V_j := \pm(V_{j\pm 1} - V_j)$, and the numerical flux has the form

$$\bar{\mathcal{F}}^n_{j-1/2} = \bar{\mathcal{F}}^n_{j+1/2}\left(U^n_j, U^n_{j-1}\right) = \begin{cases} \bar{g}(U^n_j, U^n_{j-1}) =: \bar{g}^n_{j-1/2} & \text{for } j < 0, \\ h(U^n_0, U^n_{-1}) =: h^n & \text{for } j = 0, \\ \bar{f}(U^n_j, U^n_{j-1}) =: \bar{f}^n_{j-1/2} & \text{for } j > 0. \end{cases} \tag{4.2}$$

Away from the interface, we use the standard EO flux (2.9) (with obvious substitutions for u_{L}, u_{R} and q). To define the interface flux $h(U^n_0, U^n_{-1})$, we fix a time level, say n, and set $u_{\mathrm{L}} = U^n_{-1}$, $u_{\mathrm{R}} = U^n_0$. Now let (A,B) be a connection. We modify the EO flux (2.9) to capture the entropy solution of type (A,B) as follows:

$$\begin{aligned} &h^{AB}(u_{\mathrm{R}}, u_{\mathrm{L}}) = \frac{1}{2}(\tilde{f}(u_{\mathrm{R}}) + \tilde{g}(u_{\mathrm{L}})) - \frac{1}{2}\left[\int_B^{u_{\mathrm{R}}} |\tilde{f}'(w)|\,dw - \int_A^{u_{\mathrm{L}}} |\tilde{g}'(w)|\,dw\right], \\ &\tilde{f}(u) := \min\{f(u), f(B)\}, \quad \tilde{g}(u) = \min\{g(u), g(A)\}. \end{aligned} \tag{4.3}$$

We have defined an entire class of schemes, one for each connection (A,B). Since $g \in \mathrm{Lip}([0,a_{\mathrm{L}}])$, $f \in \mathrm{Lip}([0,a_{\mathrm{R}}])$, each of the numerical fluxes is also Lipschitz continuous: $\bar{g} \in \mathrm{Lip}([0,a_{\mathrm{L}}]^2)$, $\bar{f} \in \mathrm{Lip}([0,a_{\mathrm{R}}]^2)$, $h^{AB} \in \mathrm{Lip}([0,a_{\mathrm{R}}] \times [0,a_{\mathrm{L}}])$.

5. Analysis of the scheme and convergence

We now establish assorted stability properties of the finite difference scheme and prove that it converges to an entropy solution of type (A,B) as the grid size tends to zero. We suppress dependence on the connection (A,B) to simplify notation, but we emphasize that we are discussing an entire class of schemes, one for each connection (A,B). In addition to (2.2), for the convergence analysis—but not in the statement of the main result, Theorem 5.6—we assume $u_0 \in BV(\mathbb{R})$. Thanks to Theorem 3.5 there is no loss of generality in doing so. For simplicity, we assume moreover that u_0 is compactly supported, which implies that all subsequent sums over j are finite. To deal with the general case, we can again use Theorem 3.5.

In what follows we will assume that the discretization parameters $\Delta := (\Delta x, \Delta t)$ are chosen so that the following CFL condition holds:

$$\lambda \max_{u \in [0,a_{\mathrm{R}}]} |f'(u)| \leq 1, \quad \lambda \max_{u \in [0,a_{\mathrm{L}}]} |g'(u)| \leq 1. \tag{5.1}$$

The proofs of the following two lemmas rely on standard arguments, see [**7**].

LEMMA 5.1. *The values $\{U^n_j\}$ satisfy a discrete version of* $((\mathbf{D}.1))$*: $U^n_j \in [0,a_{\mathrm{L}}]$ for $j \leq 0$, and $U^n_j \in [0,a_{\mathrm{R}}]$ for $j > 0$. Moreover, the scheme* (4.1) *is monotone.*

LEMMA 5.2. *There exists a constant C, depending on* $\mathrm{TV}(u_0)$, *but independent of Δ and n, such that $\Delta x \sum_{j\in\mathbb{Z}} |U^{n+1}_j - U^n_j| \leq \Delta x \sum_{j\in\mathbb{Z}} |U^1_j - U^0_j| \leq C\Delta t$.*

To establish compactness, we a need a spatial variation bound. Let $V^b_a(z)$ denote the total variation of $x \mapsto z(x)$ over $[a,b]$. The following lemma is essentially Lemma 4.2 of [**4**], where a proof can be found.

LEMMA 5.3. *Let $\{\xi_1, \dots, \xi_M\} \subset \mathbb{R}$ be fixed. Suppose that U^n_j is generated by an algorithm whose incremental form is $U^{n+1}_j = U^n_j + C^n_{j+1/2}\Delta_+ U^n_j - D^n_{j-1/2}\Delta_- U^n_j$, except at a finite number of indices j with $|x_j - \xi_m| \leq \rho\Delta x$ for some $m = 1, \dots, M$, where $\rho > 0$. Assume $C^n_{j+1/2} \geq 0$, $D^n_{j+1/2} \geq 0$ and $C^n_{j+1/2} + D^n_{j+1/2} \leq 1$. Finally, assume that the approximations U^n_j satisfy a time-continuity estimate of the form*

given in Lemma 5.2. Then for any interval $[a,b]$ such that $\{\xi_1,\dots,\xi_M\}\cap[a,b]=\varnothing$, and any $t\in\bar{\mathcal{T}}$ we have a spatial variation bound of the form

$$V_a^b(u^\Delta(\cdot,t))\le C(a,b), \tag{5.2}$$

where $C(a,b)$ is independent of Δ and t for $t\in\bar{\mathcal{T}}$.

Using the incremental form of our scheme, one can then prove the following lemma (see [**7**]).

LEMMA 5.4. *For any interval $[a,b]\not\ni 0$ and any $t\in\bar{\mathcal{T}}$ we have a spatial variation bound of the form* (5.2)*, where $C(a,b)$ is independent of Δ and t for $t\in\bar{\mathcal{T}}$.*

If the interface flux is not involved, the discrete entropy inequalities

$$\begin{aligned} |U_j^{n+1}-c| &\le |U_j^n-c|-\lambda\Delta_-\bar{F}_{j+1/2}^n \qquad \text{for } j\ge 1,\\ |U_j^{n+1}-c| &\le |U_j^n-c|-\lambda\Delta_-\bar{G}_{j+1/2}^n \qquad \text{for } j< -1 \end{aligned} \tag{5.3}$$

hold for any $c\in\mathbb{R}$. Here $\bar{F}_{j+1/2}^n$ and $\bar{G}_{j+1/2}^n$ are discrete entropy fluxes defined by

$$\bar{Q}_{j+1/2}^n := \bar{q}(U_{j+1}^n\vee c, U_j^n\vee c)-\bar{q}(U_{j+1}^n\wedge c, U_j^n\wedge c), \tag{5.4}$$

where $\bar{Q}$ denotes either $\bar{F}$ or $\bar{G}$ and $\bar{q}$ denotes either $\bar{f}$ or $\bar{g}$. The entropy inequalities (5.3) with the discrete entropy flux (5.4) are standard for monotone schemes. We discretize the function $c^{AB}(x)$ by $c_j:=A$ for $j<0$ and $c_j:=B$ for $j\ge 0$.

In addition to (5.3), we have the following discrete "adapted" entropy inequality. Its proof is based on the Crandall-Majda argument, see [**7**] for details.

LEMMA 5.5. *The approximate solutions U_j^n satisfy the cell entropy inequality $|U_j^{n+1}-c_j|\le|U_j^n-c_j|-\lambda\Delta_-\mathcal{H}_{j+1/2}^n$. Here, the numerical entropy flux is given by*

$$\begin{aligned}\mathcal{H}_{j-1/2}^n &= \mathcal{F}_{j-1/2}\left(U_j^n\vee c_j, U_{j-1}^n\vee c_{j-1}\right)-\mathcal{F}_{j-1/2}\left(U_j^n\wedge c_j, U_{j-1}^n\wedge c_{j-1}\right)\\ &=\begin{cases}\bar{g}\left(U_j^n\vee A, U_{j-1}^n\vee A\right)-\bar{g}\left(U_j^n\wedge A, U_{j-1}^n\wedge A\right) & \text{for } j<0,\\ h\left(U_0^n\vee B, U_{-1}^n\vee A\right)-h\left(U_0^n\wedge B, U_{-1}^n\wedge A\right) & \text{for } j=0,\\ \bar{f}\left(U_j^n\vee B, U_{j-1}^n\vee B\right)-\bar{f}\left(U_j^n\wedge B, U_{j-1}^n\wedge B\right) & \text{for } j>0.\end{cases}\end{aligned}$$

We can now state our main theorem.

THEOREM 5.6. *Suppose* (2.1) *and* (2.2) *hold. Let (A,B) be a connection, and u^Δ be defined as in Sect. 4 for the (A,B)-version of the scheme* (4.1)–(4.3)*. Let $\Delta\to 0$ with $\lambda=\Delta x/\Delta t$ constant and the CFL condition* (5.1) *satisfied. Then there exists a function u such that $u^\Delta\to u$ in $L^1_{\mathrm{loc}}(\Pi_T)$ and a.e. in Π_T. The limit function u is an entropy solution of type (A,B) of* (1.1)*. In particular, there exists a (unique) entropy solution of type (A,B) to the initial value problem* (1.1)*.*

SKETCH OF PROOF. We assume that $u_0\in BV(\mathbb{R})$. The general case (2.2) then follows from the L^1 contractivity (Theorem 3.5). For our approximate solutions u^Δ we have an L^∞ bound (Lemma 5.1), a time continuity bound (Lemma 5.2), and a bound on the spatial variation in any interval $[a,b]\not\ni 0$. By standard compactness results, for any fixed interval $[a,b]\not\ni 0$, there is a subsequence (which we do not relabel) such that u^Δ converges in $L^1([a,b]\times\bar{\mathcal{T}})$. Taking a countable set of intervals $[a_i,b_i]$ such that $\bigcup_i[a_i,b_i]=\mathbb{R}\setminus\{0\}$, and employing a standard diagonal process we can extract a subsequence (which we again do not relabel) such that u^Δ converges in $L^1_{\mathrm{loc}}(\Pi_T)$ and also a.e. in Π_T to some $u\in L^\infty(\Pi_T)$ with $u(x,t)$ satisfying (**D**.1).

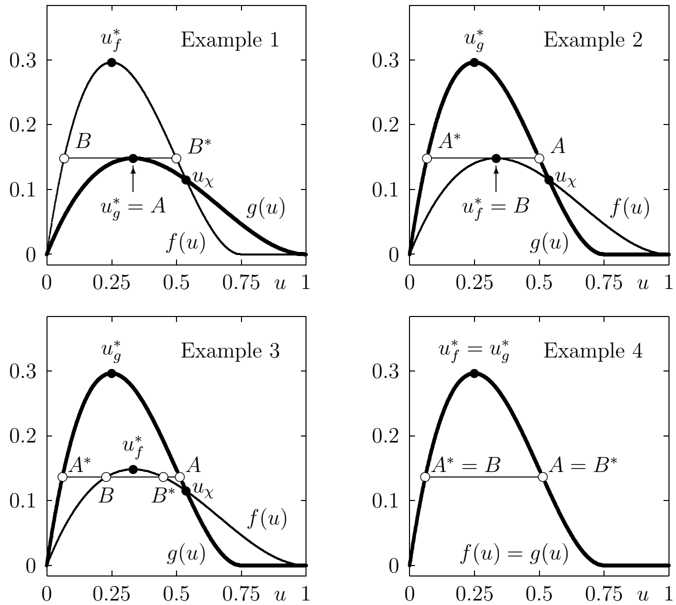

FIGURE 1. Fluxes $f(u)$ and $g(u)$, extrema u_f^* and u_g^* and connection (A, B) for Examples 1 to 4. The thin horizontal line marks the chosen connection, which limits the flux across $x = 0$.

Although not required from Definition 3.1, it follows from the time continuity estimate (5.2) that $u \in C(0, T; L^1(\mathbb{R}))$. Additionally, the initial data u_0 is taken by u in the strong L^1_{loc} sense. The remaining parts of Definition 3.1 are verified by several Lax-Wendroff-type calculations applied to (5.3), see [**7**] for details. $\square$

6. Numerical examples

We illustrate the scheme by four examples of cases not included in our previous work [**5, 7**], utilizing fluxes given by $\psi(u) := \frac{8}{3}u(1 - \frac{4}{3}u)^2$ for $u \in [0, \frac{3}{4}]$, $\psi(u) = 0$ otherwise, $\varphi(u) := u(1 - u)^2$ for $u \in [0, 1]$ and $\varphi(u) = 0$ otherwise. The functions ψ and φ intersect at $u_\chi \approx 0.5377$. In Example 1, we set $f(u) = \psi(u)$, $g(u) = \varphi(u)$, and choose the flux-maximizing connection (A, B) with $A = u_g^* = \frac{1}{3}$ and the corresponding value $B = (2 - \sqrt{3})/4 \approx 0.06699$. (Fig. 1 illustrates the fluxes of the examples.) In Example 1, $B^* = 0.5$ is the endpoint of the horizontal segment BB^* which limits the flux across $x = 0$. In Example 2, we set $f(u) = \varphi(u)$ and $g(u) = \psi(u)$, and choose again the flux-maximizing connection with $B = u_f^* = \frac{1}{3}$ and the corresponding values $A = 0.5$ and $A^* = 0.06699$. In Example 3, we choose f and g as in Example 2, but decide to use a connection (A, B) which limits the flux across $x = 0$ by $g(A) = f(B) = g(A^*) = f(B^*) = \varphi(0.45)$. Finally, in Example 4 we utilize $f(u) = g(u) = \psi(u)$, but limit the flux across $x = 0$ by the value $\varphi(0.45)$. This corresponds to a connection of the type $(A = B^*, B = A^*)$. In all cases we choose the initial datum $u_0(x) = 1$ (Example 1) and $u_0(x) = 0.75$ (Examples 2–4) for $x \in [-1, 0]$, and $u_0(x) = 0$ otherwise, along with $\Delta x = 1/50$ and $\lambda = 0.375$.

Fig. 2 (a) shows the numerical solution for Example 1, which exibits a downward-facing jump between A and B. This solution may be compared with that of the standard conservation law with $f(u) = g(u) = \varphi(u)$ (Fig. 2 (b)), which in terms

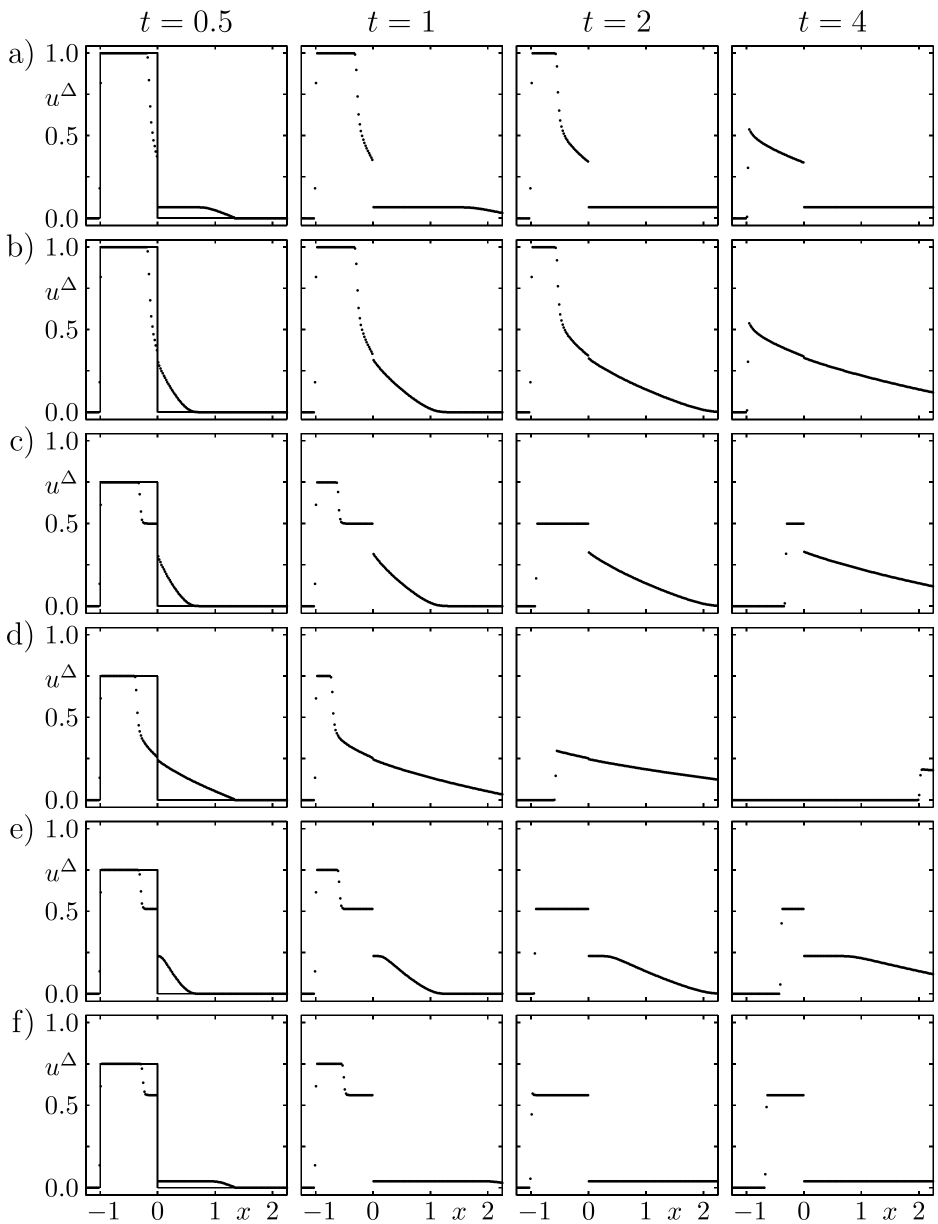

FIGURE 2. Numerical solutions for (a) Example 1, (b) $f(u) = g(u) = \varphi(u)$, (c) Example 2, (d) $f(u) = g(u) = \psi(u)$, (e) Example 3 and (f) Example 4. The thin lines in the plots for $t = 0.5$ mark the initial concentration.

of the traffic model shows that here the flux transition accelerates the traffic flow. The basic idea behind Example 2 is to interchange the roles of f and g. Fig. 2 (c) displays the numerical solution, and comparison with the case $f(u) = g(u) = \psi(u)$ (Fig. 2 (d)) shows that now the change of fluxes at $x = 0$ acts at a bottleneck. Example 3 produces a solution (see Fig. 2 (e)) similar to Example 2, but which

leads to solutions that have different discontinuities across $x = 0$. Finally, Example 4 (Fig. 2 (f)) illustrates the effect of limiting the flux at just one single point. Here, the solution is markedly different from Fig. 2 (d), although different numerical fluxes are used at one single point only.

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