On Conservation Laws with Discontinuous Flux

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1 Introduction

In this contribution we are interested in spatially one-dimensional conservation laws

$$u_t + \mathcal{F}\big(\boldsymbol{\gamma}(x), u\big)_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{1}$$

that depend discontinuously on the location x via a vector $\boldsymbol{\gamma}(x)$. Specifically, we consider problems of two different conservation laws defined on adjacent domains:

$$u_t + f(u)_x = 0 \quad \text{for } x > 0, \quad u_t + g(u)_x = 0 \quad \text{for } x < 0,\ t > 0. \tag{2}$$

Applications of conservation laws involving such a discontinuous flux include models of two-phase flow in porous media [11,14], traffic flow with discontinuous road surface [3,13], shape-from-shading problems [15], and clarifier-thickener models of continuous sedimentation [2,4–9]. The basic difficulty for the mathematical and numerical analysis of these models consists in the identification of an admissibility criterion a jump of the solution has to satisfy across a discontinuity of the flux.

In this note, we discuss two alternative entropy solution concepts for conservation laws with a flux function that depends discontinuously on the space variable. It is demonstrated that two different applications, namely clarifier-thickener models and models of two-phase flow in heterogeneous porous media, lead to two different entropy jump conditions.

This paper is organized as follows. In Sections 2 and 3, we introduce the clarifier-thickener and two-phase flow in porous media model, respectively. The corresponding entropy solution concepts and numerical schemes are introduced in Sections 4.1 and 4.2, respectively. In Section 5 we illustrate the differences between the entropy solution approaches to the two models by numerical examples. Section 6 provides a concluding discussion.

2 Clarifier-Thickener Model

Under idealizing assumptions, the gravity settling of small, equal-sized solid particles in a viscous fluid can be described by Kynch's one-dimensional kinematic sedimentation model. Its main assumption states that if the suspension is considered as a superposition of two continuous phases, then the solid-fluid relative or slip velocity v_r is a material specific function of the local solids concentration u only. The governing equation is the conservation law $u_t + (q(x,t)u + u(1-u)v_\mathrm{r}(u))_x = 0$, where $q = q(x,t)$ is the volume average velocity of the mixture, t is time and x is depth. In absence of sources and sinks, we have $q = q(t)$. Introducing the function $b(u) := u(1-u)v_\mathrm{r}(u)$, we can rewrite the governing equation as

$$\partial_t u + \partial_x\big(q(x,t)u + b(u)\big) = 0. \tag{3}$$

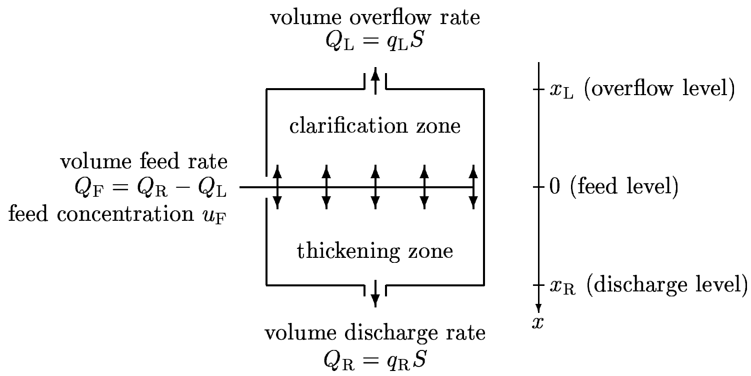


Fig. 1. The one-dimensional clarifier-thickener model.

The function $b(u)$ is assumed to be Lipschitz continuous, positive for $u \in (0,1)$, and to vanish for $u \notin (0,1)$. We assume that $b \in C^2(0,1)$, that $b'(u)$ vanishes at exactly one location $u = u_{\max} \in (0,1)$, where the function has a maximum, and that $b''(u)$ vanishes at no more than one inflection point in $u_{\text{infl}} \in (0,1)$; if such a point is present, we assume that $u_{\text{infl}} \in (u_{\max}, 1)$. These assumptions are valid for the frequently used function

$$b(u) = \begin{cases} v_\infty u(1-u)^n & \text{for } u \in (0,1), \\ 0 & \text{for } u \leq 0 \text{ or } u \geq 1, \end{cases} \quad v_\infty > 0,\ n \geq 1, \tag{4}$$

where v_∞ denotes the settling velocity of a single particle in an unbounded pure fluid.

Consider now the ideal clarifier-thickener model sketched in Figure 1. This is an idealized cylindrical vessel of cross-sectional area S occupying a vertical interval $x \in [x_L, x_R]$. At $x = 0$, a feed source is located, through which fresh suspension is pumped into the vessel (and distributed over the entire cross-sectional area) at a constant volume rate $Q_F \geq 0$. This induces an upward mixture flow for $x < 0$ and a downward mixture flow for $x > 0$. The vessel is equipped with a controllable discharge opening at $x = x_R$ and a controllable overflow outlet at $x = x_L$. The control variables are the volume overflow rate $Q_L \leq 0$ and the volume discharge rate $Q_R \geq 0$, where $Q_F = Q_R - Q_L$. We define the volume average velocities $q_R = Q_R/S \geq 0$ and $q_L = Q_L/S \leq 0$. Thus, the volume average velocity $q(x,t)$ is

$$q(x,t) = q_L \leq 0 \quad \text{for } x < 0, \quad q(x,t) = q_R \geq 0 \quad \text{for } x > 0. \tag{5}$$

We assume that the mixture leaving the unit at $x = x_R$ and $x = x_L$ is transported away at the velocities q_R and q_L, respectively, through a thin pipe in which the solids and the fluid move with the same velocity. Thus, the slip velocity v_r vanishes (or equivalently, the flux $b(u)$ is not present) outside $[x_L, x_R]$.

Next, we have to consider that $x = 0$ solid material of a given concentration u_F is fed into the unit. This means that the zero right-hand part of the governing conservation law has to be replaced by the singular source term $\delta(x) Q_F u_F$, where $\delta(\cdot)$ is the Dirac unit mass located at $x = 0$. However, using the Heaviside function $H(x)$, we may formally write

$$\frac{\delta(x) Q_F u_F}{S} = \delta(x)(q_R - q_L)u_F = \partial_x \big(H(x)(q_R - q_L)u_F \big),$$

and thereby express the singular source as a discontinuity of the flux function.

Finally, assuming that an initial concentration $u_0(x)$ for $x \in \mathbb{R}$ is given (for example, $u_0 \equiv 0$ for a system that initially contains only water), we can model the clarifier-thickener

unit as the following conservation law for the unknown concentration $u(x,t)$:

$$\begin{cases} \partial_t u + \partial_x g(x,u) = 0, & (x,t) \in \Pi_T := \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

where $T > 0$ is fixed, and the discontinuously spatially varying flux g has the form

$$g(x,u) = \begin{cases} q_{\mathrm{L}}(u-u_{\mathrm{F}}) & \text{for } x < -1, \qquad q_{\mathrm{R}}(u-u_{\mathrm{F}}) + b(u) \quad \text{for } 0 < x < 1, \\ q_{\mathrm{L}}(u-u_{\mathrm{F}}) + b(u) & \text{for } -1 < x < 0, \quad q_{\mathrm{R}}(u-u_{\mathrm{F}}) \qquad\qquad \text{for } x > 1. \end{cases} \tag{7}$$

In this work, we assume that the control variables q_{L}, q_{R}, and u_{F} are constant in time.

3 Two-phase flow in porous media

Capillary-free two-phase incompressible flow in a porous medium with 'rock type' changing abruptly at $x = 0$ can be modeled by the pair of conservation laws (2), where u is the saturation of one of the phases, say phase 1, and assumes values in an interval $[s,S]$. The functions $f(u)$ and $g(u)$ are the Darcy velocities of phase 1 in each rock type, and they have the forms

$$f(u) = \frac{\lambda_1(q + (c_1 - c_2)\lambda_2)}{\phi(\lambda_1 + \lambda_2)}, \quad g(u) = \frac{\mu_1(q + (c_1 - c_2)\mu_2)}{\phi(\mu_1 + \mu_2)}, \tag{8}$$

where ϕ is the porosity of the rock and q, a constant in space, is the total Darcy velocity, that is, the sum of the Darcy velocities of the two phases. The quantities λ_1, μ_1 and λ_2, μ_2 are the effective mobilities of the two phases, where λ_1 and μ_1 are increasing functions of u, λ_2 and μ_2 are decreasing functions of u, $\lambda_1(s) = \mu_1(s) = 0$, and $\lambda_2(S) = \mu_2(S) = 0$. The gravity constants c_1 and c_2 are proportional to the density of phase 1 and 2, respectively.

4 Solution concepts and jump conditions

4.1 Clarifier-thickener and related models

To outline the mathematical theory for the clarifier-thickener model, consider the conservation law (1), where the flux is $\mathcal{F}(\boldsymbol{\gamma}(x),u) = g(x,u) = \gamma_1(x)(u-u_{\mathrm{F}}) + \gamma_2(x)b(u)$ with

$$\boldsymbol{\gamma}(x) := (\gamma_1(x), \gamma_2(x)), \quad \gamma_1(x) := \begin{cases} q_{\mathrm{L}} & \text{if } x < 0, \\ q_{\mathrm{R}} & \text{if } x > 0, \end{cases} \quad \gamma_2(x) := \begin{cases} 1 & \text{if } x \in (x_{\mathrm{L}}, x_{\mathrm{R}}), \\ 0 & \text{otherwise.} \end{cases}$$

Independently of the smoothness of $\boldsymbol{\gamma}(x)$, solutions to (1) are in general not smooth and weak solutions must be sought, that is, integrable functions $u(x,t)$ taking values in $[0,1]$ such that $u(t) \to u_0$ as $t \to 0+$ (u_0 is the prescribed initial function), and (1) is satisfied in the sense of distributions. However, discontinuous weak solutions are in general not uniquely determined by their initial data, so that a so-called entropy condition must be imposed to single out the physically correct solution or entropy weak solution.

Suppose for the moment that $\boldsymbol{\gamma}(x) = (\gamma_1, \gamma_2)$ is smooth. Then a weak solution u is said to satisfy the *entropy condition* if for all convex functions $\eta \in C^2(\mathbb{R})$, there holds

$$\eta(u)_t + F\big(\boldsymbol{\gamma}(x),u\big)_x + \boldsymbol{\gamma}'(x) \cdot \big(\eta'(u)\mathcal{F}_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}(x),u) - F_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}(x),u)\big) \leq 0 \tag{9}$$

in the sense of distributions, where the entropy flux $F(\boldsymbol{\gamma},u)$ is defined by $F_u(\boldsymbol{\gamma},u) := \eta'(u)\mathcal{F}_u(\boldsymbol{\gamma},u)$. Formally, (9) is obtained by multiplying the regularized parabolic equation

$$u_t^\varepsilon + \mathcal{F}\big(\boldsymbol{\gamma}(x),u^\varepsilon\big)_x = \varepsilon u_{xx}^\varepsilon, \tag{10}$$

where $\varepsilon > 0$ is a small regularization parameter, by $\eta'(u)$, using the chain rule, discarding the parabolic dissipation term $\eta''(u)\varepsilon u_{xx}$ thanks to the convexity of η, and letting $\varepsilon \downarrow 0$.

By a standard approximation argument, (9) implies the entropy condition

$$\partial_t|u-c| + \big(\operatorname{sgn}(u-c)\big[\mathcal{F}(\boldsymbol{\gamma}(x),u) - \mathcal{F}(\boldsymbol{\gamma}(x),c)\big]\big)_x + \operatorname{sgn}(u-c)\boldsymbol{\gamma}'(x)\cdot\mathcal{F}_{\boldsymbol{\gamma}}\big(\boldsymbol{\gamma}(x),c\big) \le 0$$

for all $c \in [0,1]$ in the sense of distributions. When $\boldsymbol{\gamma}$ is smooth, it is well known that (1) has a unique and stable entropy weak solution.

The very notion of entropy weak solution introduced above (in particular the entropy condition) and the corresponding well-posedness theory are not applicable when $\boldsymbol{\gamma}$ is discontinuous. In [6] we suggest the following variant of the above notion of entropy weak solution that accounts for the discontinuities in $\boldsymbol{\gamma}$.

Let $\mathcal{J} := \{x_{\mathrm{L}}, 0, x_{\mathrm{R}}\}$ denote the set points where $\boldsymbol{\gamma}$ is discontinuous, and for a point $m \in \mathcal{J}$, we use the notation $\boldsymbol{\gamma}(m-)$ and $\boldsymbol{\gamma}(m+)$ for the one-sided limits at m. Then we say that a function $u(x,t)$ is a BV_t *entropy weak solution* of the initial value problem for (1) if it satisfies the following conditions:

1. *Regularity.* The function u belongs to $L^1 \cap BV_t$ and $u(x,t) \in [0,u_{\max}]$ for all (x,t).
2. *Weak formulation.* For all test functions $\phi(x,t)$, we have

$$\int_0^T \int_{-\infty}^{\infty} \big(u\phi_t + \mathcal{F}\big(\boldsymbol{\gamma}(x),u\big)\phi_x\big)\,dx\,dt = 0. \tag{11}$$

3. *Initial condition.* The initial condition is satisfied in the following strong sense:

$$\lim_{t\downarrow 0}\int_{-\infty}^{\infty}\big|u(x,t)-u_0(x)\big|\,dx = 0. \tag{12}$$

4. *Entropy condition.* The following entropy inequality holds for all $c \in [0,1]$ and all non-negative test functions $\phi(x,t)$:

$$\begin{aligned}\int_0^T \int_{-\infty}^{\infty} \Big(|u-c|\phi_t + \operatorname{sgn}(u-c)\Big(\mathcal{F}\big(\boldsymbol{\gamma}(x),u\big) - \mathcal{F}\big(\boldsymbol{\gamma}(x),c\big)\Big)\phi_x\Big)\,dx\,dt&\\ + \int_0^T \sum_{m\in\mathcal{J}} \big|\mathcal{F}\big(\boldsymbol{\gamma}(m+),c\big) - \mathcal{F}\big(\boldsymbol{\gamma}(m-),c\big)\big|\phi(m,t)\,dt \ge 0.&\end{aligned} \tag{13}$$

A function $u(x,t)$ satisfying only conditions (1)–(3) is called a BV_t *weak solution* of the initial value problem for (1).

Following [12], we proved in [6] that BV_t entropy weak solutions as defined above are unique and depend continuously in L^1 on their initial values. More precisely, we proved the following statement: Let v and u be two BV_t entropy weak solutions to the initial value problem for (1). Then for any $t \in (0,T)$

$$\big\|v(\cdot,t)-u(\cdot,t)\big\|_{L^1} \le \big\|v(\cdot,0)-u(\cdot,0)\big\|_{L^1}. \tag{14}$$

Among many things, the proof of (14), which immediately implies uniqueness, relies on jump conditions that relate limits from the right and left of the BV_t entropy weak solution

u at jumps in the spatially varying coefficient $\gamma(x)$. More specifically, we use a Rankine-Hugoniot condition expressing conservation across each jump, which is a consequence of (11), and also an entropy jump condition, which is a consequence of (13).

Let u be a BV_t entropy weak solution. Fix one of the jumps in $m \in \mathcal{J}$. Then the following Rankine-Hugoniot condition holds across the jump for a.e. $t \in (0,T)$:

$$\mathcal{F}(\boldsymbol{\gamma}_+, u_+) = \mathcal{F}(\boldsymbol{\gamma}_-, u_-), \tag{15}$$

where "−" and "+" denote spatial limits from the left and right of the jump, respectively. Furthermore, for $u_-(t) \neq u_+(t)$, the following entropy jump condition holds:

$$F\big(\boldsymbol{\gamma}_+, u_+, c\big) - F\big(\boldsymbol{\gamma}_-, u_-, c\big) \leq \big|\mathcal{F}(\boldsymbol{\gamma}_+, c) - \mathcal{F}(\boldsymbol{\gamma}_-, c)\big| \quad \text{for all } c \in [0,1],$$

where $F(\boldsymbol{\gamma}, u, c) = \operatorname{sgn}(u-c)\big[\mathcal{F}(\boldsymbol{\gamma}, u) - \mathcal{F}(\boldsymbol{\gamma}, c)\big]$.

A working upwind difference scheme for generating approximate solutions to the clarifier thickener model can be stated as follows. We discretize the spatial domain $\mathbb{R}$ into cells $I_j := [x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$, where $x_k = k\Delta x$ for $k = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$. Similarly, the time interval $(0,T)$ is is discretized via $t^n = n\Delta t$ for $n = 0, \dots, N$, where $N = \lfloor T/\Delta t \rfloor + 1$, which results in the time strips $I^n := [t^n, t^{n+1})$, $n = 0, \dots, N-1$. Here $\Delta x > 0$ and $\Delta t > 0$ denote the spatial and temporal discretization parameters, respectively. The discretization parameters are chosen so that the following CFL stability condition holds:

$$\lambda\big(\max\{-q_{\mathrm{L}}, q_{\mathrm{R}}\} + \|h'\|\big) \leq 1/2, \qquad \lambda := \Delta t/\Delta x. \tag{16}$$

When sending $\Delta \downarrow 0$, the ratio λ will be kept constant. We denote by U_j^n the finite difference approximation of $u(j\Delta x, n\Delta t)$. The initial data are discretized by

$$U_j^0 := \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \qquad j \in \mathbb{Z}, \tag{17}$$

while the discretization of $\boldsymbol{\gamma}(x)$ is *staggered* with respect to that of u:

$$\boldsymbol{\gamma}_{j+1/2} := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \boldsymbol{\gamma}(x)\,dx, \qquad j \in \mathbb{Z}. \tag{18}$$

By staggering these discretizations we can avoid the 2×2 Riemann problems (as in [4]) that arise when the discretizations are aligned. The result is that our algorithm is simple to implement. Indeed, we compute $\{U_j^n\}$ by the explicit difference scheme

$$U_j^{n+1} = U_j^n - \lambda\big[h\big(\boldsymbol{\gamma}_{j+1/2}, U_{j+1}^n, U_j^n\big) - h\big(\boldsymbol{\gamma}_{j-1/2}, U_j^n, U_{j-1}^n\big)\big],\ j \in \mathbb{Z},\ n = 0, \dots, N-2. \tag{19}$$

Here, $h(\boldsymbol{\gamma}, v, u)$ is the Engquist-Osher (EO) numerical flux [10]

$$h(\boldsymbol{\gamma}, v, u) := \frac{1}{2}\big(\mathcal{F}(\boldsymbol{\gamma}, u) + \mathcal{F}(\boldsymbol{\gamma}, v)\big) - \frac{1}{2}\int_u^v |\mathcal{F}_u(\boldsymbol{\gamma}, w)|\,dw. \tag{20}$$

The EO numerical flux is a consistent, two-point monotone flux that is Lipschitz continuous with respect to each of its arguments. With the EO flux (20), the resulting algorithm is a so-called upwind scheme, i.e., the differencing of the flux is biased in the direction of incoming waves. This allows resolving shocks without excessive smearing. The choice of the EO flux is also motivated by its close functional relationship to the Kružkov entropy

flux and a particular nonlinear singular mapping. These relationships are used to prove compactness for the sequence of numerical approximations $\{u^{\Delta}\}$.

Thanks to its upwind nature, the scheme (17)–(20) reproduces within reasonable accuracy the discontinuities in the solutions without the necessity to track them explicitly, i.e., it has the shock-capturing property. An obvious requirement of any numerical scheme is that it should approximate (converge to) the correct solution of the problem it is trying to solve. In particular, in the present context a numerical scheme should converge as the discretization parameters tend to zero to a limit function that satisfies the entropy condition (13). This implies that the scheme produces solutions with correct discontinuities. It was proved in [6] that under some technical conditions on the initial function u_0 and assuming that the CFL condition (16) holds, the scheme (17)–(20) has this desired property.

4.2 Two-phase Flow in Porous Media

In [1], Adimurthi, Jaffré and Veerappa Gowda utilize a slightly different concept of entropy solutions for equation (2), which is based on a standard 'interior' entropy condition valid away from the flux discontinuity at $x = 0$ combined with a particular 'interface' entropy condition that is designed to prevent undercompressive shocks emerging from the flux discontinuity. This entropy concept can be stated as follows. We require the interior condition

$$\begin{aligned} &\forall\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\} \times \mathbb{R}^+), \varphi \geq 0 : \ \forall c \in \mathbb{R} : \\ &\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big\{ |u-c|\varphi_t + \operatorname{sgn}(u-c)\Big(\mathcal{F}\big(\gamma(x),u\big) - \mathcal{F}\big(\gamma(x),c\big)\Big)\varphi_x \Big\}\, dx\, dt \geq 0, \end{aligned} \tag{21}$$

which is supplemented by the interface entropy condition. To state it, let us assume, for simplicity, the special situation where the flux functions $f(u)$ and $g(u)$ are assumed to be differentiable with compact support on the interval $[0,1]$ with $f(0) = g(0) = f(1) = g(1) = 0$, $f(u) > 0$ and $g(u) > 0$ on $(0,1)$, and having exactly one maximum θ_f and θ_g on $(0,1)$ with $\theta_g < \theta_f$, $f'(u) \neq 0$ for $u \in (0,\theta_f) \cup (\theta_f, 1)$ and $g'(u) \neq 0$ for $u \in (0,\theta_g) \cup (\theta_g,1)$. Moreover, these functions are assumed to intersect in exactly one point, for simplicity again called u_{F}, on $(0,1)$. Observe that in the vicinity of u_{F}, we have $g'(u) < 0$ for $u \leq u_{\mathrm{F}}$ and $f'(u) > 0$ for $u \geq u_{\mathrm{F}}$. Now, assume that u is a weak solution of (2) such that the limits $u_+(t) := \lim_{x\to 0^+} u(x,t)$ and $u_-(t) := \lim_{x \to 0^-} u(x,t)$ exist for almost all $t > 0$. Firstly, we require, of course, that the jump condition (15) be satisfied, which here means that

$$f\big(u_+(t)\big) = g(u_-(t)) \quad \text{for almost all } t. \tag{22}$$

To state the interface condition, we define the sets

$$\begin{aligned} L &:= \big\{ t > 0 : \ u_+(t) \in [0,\theta_f) \wedge u_-(t) \in (\theta_g, 1] \big\}, \\ U &:= \big\{ t \in L : \ u_+(t) = u_-(t) = 0 \vee u_+(t) = u_-(t) = 1 \big\}; \end{aligned}$$

then u is said to satisfy the interface condition if $\operatorname{meas}(L\backslash U) = 0$. Note that this condition excludes undercompressive jumps across $x = 0$, which are defined by $f'(u_+(t)) > 0$ and $g'(u_-(t)) < 0$, and for which the characteristics would emerge from $x = 0$ to both sides. In particular, although a solution with $u_-(t) = u_+(t) = u_{\mathrm{F}}$ does satisfy the jump condition (22), the constant $u \equiv u_{\mathrm{F}}$ is not an entropy solution since the interface condition is violated.

Adimurthi et al. [1] utilize the following Godunov-type scheme for the numerical solution of (2) under the assumptions on the fluxes f and g stated above. If we define the standard

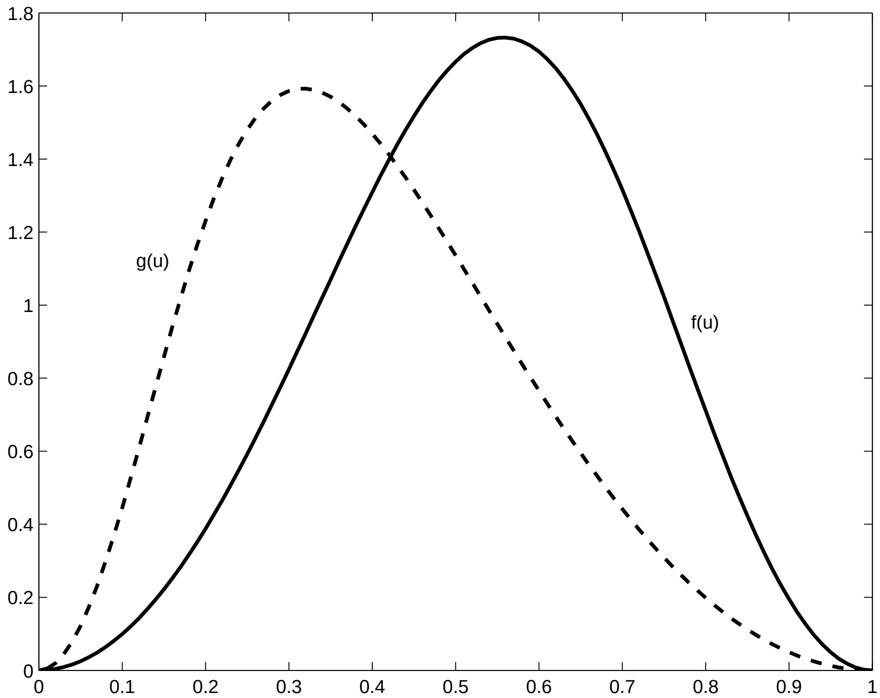


Fig. 2. Left and right flux functions $g(u)$ and $f(u)$ for the porous media model, given by (25).

Godunov numerical fluxes

$$h_f(a,b) := \begin{cases} \min_{s\in[a,b]} f(s) & \text{if } a < b, \\ \max_{s\in[b,a]} f(s) & \text{if } b \geq a, \end{cases} \qquad h_g(a,b) := \begin{cases} \min_{s\in[a,b]} g(s) & \text{if } a < b, \\ \max_{s\in[b,a]} g(s) & \text{if } b \geq a, \end{cases} \tag{23}$$

as well as the interface numerical flux $\bar{h}(a,b) := \min\{g(\min\{0,\theta_g\}), f(\max\{\theta_f,1\}\}$, the final scheme corresponds to replacing (19) by

$$U_j^{n+1} = U_j^n - \lambda \cdot \begin{cases} (h_g(U_j^n, U_{j+1}^n) - h_g(U_{j-1}^n, U_j^n)) & \text{if } j < -1, \\ (\bar{h}(U_{-1}^n, U_1^n) - h_g(U_{-2}^n, U_{-1}^n)) & \text{if } j = -1, \\ (h_f(U_1^n, U_2^n) - \bar{h}(U_{-1}^n, U_1^n)) & \text{if } j = 1, \\ (h_f(U_j^n, U_{j+1}^n) - h_f(U_{j-1}^n, U_j^n)) & \text{if } j > 1, \end{cases} \quad n = 0, \dots, N-1. \tag{24}$$

The CFL stability condition for (24) is $\lambda = \Delta t/\Delta x \leq 1/\max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}$. In [1] it is shown that the scheme converges to a weak solution of the Cauchy problem if the CFL condition is satified and the initial datum is of bounded variation (in a particular sense).

To illustrate the interface condition, we consider the following pair of functions from [1] (f,g) that satisfies all conditions stated above:

$$f(u) = \frac{20u^2(1-u)^2}{u^2 + 2(1-u)^2}, \quad g(u) = \frac{50u^2(1-u)^2}{10u^2 + (1-u)^2}, \tag{25}$$

which corresponds to (8) for $\phi = 1$, $q = 0$, $c_1 = 2$, $c_2 = 1$, $\lambda_1(u) = 10u^2$, $\lambda_2(u) = 20(1-u)^2$, $\mu_1(u) = 50u^2$ and $\mu_2(u) = 5(1-u)^2$, see Figure 2.

5 Numerical examples

In Sections 4.1 and 4.2, we introduced two alternative entropy solution concepts for the Cauchy problem of (1) or (2). Both are associated with a first-order finite difference scheme,

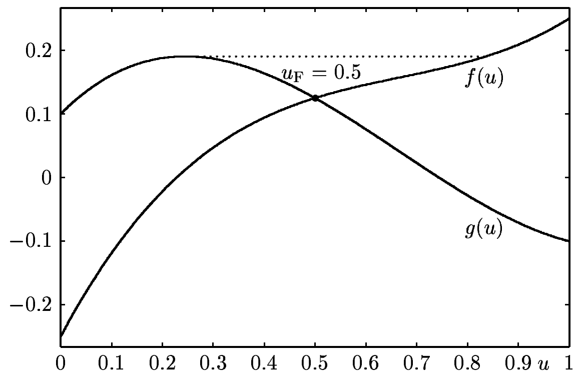


Fig. 3. Left and right flux functions $g(u)$ and $f(u)$ for the clarifier-thickener model

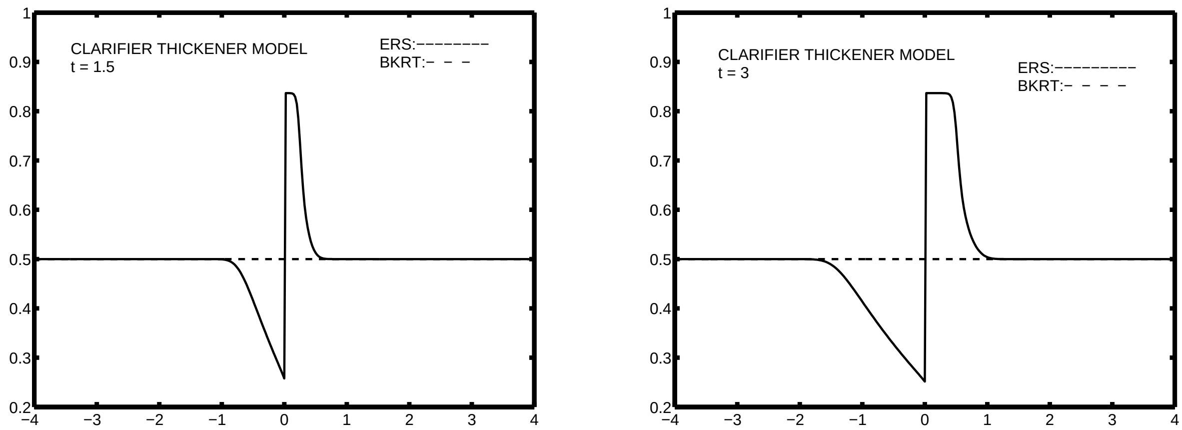
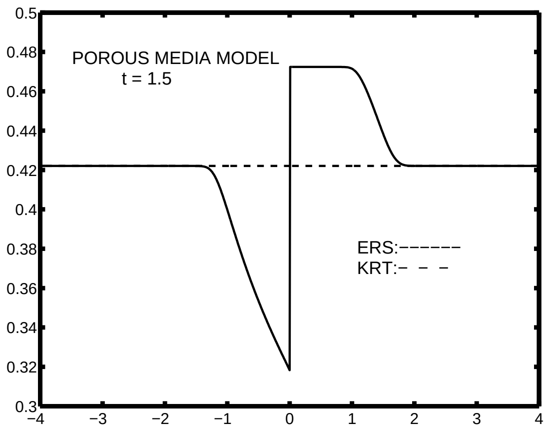


Fig. 4. Numerical simulations of the clarifier-thickener model produced both ERS and BKRT schemes

referred to as 'BKRT' scheme (according to [7]) for the scheme of Section 4.1 and as 'ERS' scheme (based on an *Exact Riemann Solver*) for that of Section 4.2, respectively. We demonstrate the differences between these concepts by applying both schemes to one test case of the clarifier-thickener model and one test case of the two-phase flow in porous media model, respectively.

We consider the clarifiier-thickener model with $b(u) = u(1-u)^2$, $q_{\mathrm{L}} = -0.2$, $q_{\mathrm{R}} = 0.5$ and u_{F}. We choose the global initial datum $u_0 = u_{\mathrm{F}} = 0.5$ and are interested in the solution near $x = 0$ and for small times only. This situation can be achieved by assuming $-x_{\mathrm{L}}$ and x_{R} sufficiently large. Thus, instead of (7) we consider a single discontinuity separating the left and right fluxes $f(u) = g(x,u)|_{(0,x_{\mathrm{R}})}$ and $g(u) = g(x,u)|_{(x_{\mathrm{L}},0)}$, which are plotted in Figure 3. Figure 4 shows the numerical results at simulated times $t = 1.5$ and $t = 3$, obtained by applying both schemes with $\Delta x = 0.025$ and $\lambda = 0.125$. We observe that the BKRT scheme implies a constant solution assuming simply the value $u_{\mathrm{F}} = 0.5$, while the ERS scheme approximates a solution that involves a jump across $x = 0$ from $\theta_g = 0.24503$ to the value $\theta_g^* \approx 0.83$ with $f(\theta_g^*) = g(\theta_g)$.

Finally, we consider the two-phase flow in porous media model with the flux function (25), and we choose the initial datum $u_{\mathrm{F}} = 0.42206$, which is just the intersection point. Figure 5 shows the solutions at $t = 1.5$ and $t = 3$ obtained from both schemes with $\Delta x = 0.01$ and $\lambda = 1/32$. These solutions are very similar to those of Figure 4.



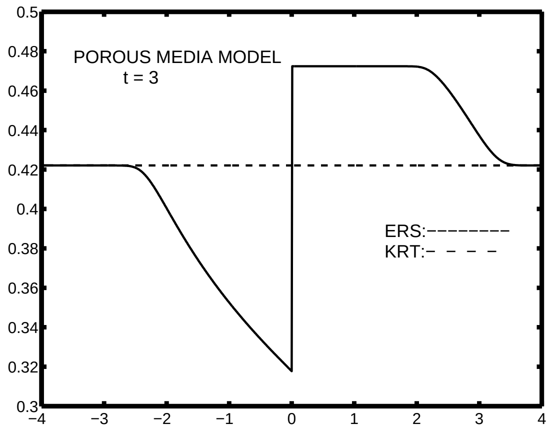


Fig. 5. Numerical simulations of the porous media model produced both ERS and BKRT schemes

6 Conclusions

The preceding examples show that the two entropy solution concepts introduced in Sections 4.1 lead to noticeably different solutions. This discrepancy can be resolved if we look again at the physical interpretation of the discontinuous flux conservation law in both models. In the C-T model, the location $x = 0$ corresponds to a source of material (namely, of the feed suspension) which is transported away to both sides of the flux discontinuity. In this situation, undercompressive solutions should not be avoided; rather, given the role of $x = 0$ as a singular material source, they appear to be the typical case here. Moreover, one intuitively expects that if suspension of a certain concentration u_{F} is injected into a tank filled with suspension of the same concentration u_{F}, then the concentration remains constant near the injection point, and the solution does not show the drastical variation produced by the ERS scheme as in Figure 4. Clearly, for the clarifier-thickener model the concept associated with the BKRT scheme is the preferable one.

Mathematically speaking, we saw that the (Kružkov) entropy formulation for the clarifier-thickener model can be derived from the parabolic regularization (10) when the regularization parameter ε tends to zero. On the other hand, models for two-phase flow in porous media are usually based on the assumption that the capillary pressure is continuous across heterogeneities of the porous medium. Consequently, the appropriate viscous regularization term of (1) for this model is not given by $\varepsilon u^{\varepsilon}_{xx}$, but by $\varepsilon(\lambda_c(u^{\varepsilon})p_c(u^{\varepsilon})_x)_x$, where λ_c and p_c are the mobility and capillary pressure functions for Phase 1 and $x < 0$ ($c = \mathrm{L}$) and $x > 0$ ($c = \mathrm{R}$), respectively [11,14]. Analyzing the limit $\varepsilon \to 0$ for this regularization term, Kaasschieter [11] shows that the corresponding viscosity limit produces an entropy condition for the limiting problem (2) that excludes that characteristics leave the discontinuity at $x = 0$ to both sides. In other words, the capillary pressure characterization does not allow undercompressive shocks emerging from $x = 0$. Therefore, the formulation of Section 4.2 is suited for two-phase flow in porous media problems.

In this note, both approaches have been presented opposed to each other. Our current goal is to find a unified well-posedness framework for discontinuous flux problems that admits a certain flexibility in incorporating different entropy jump conditions (in particular, the conditions presented here) for different physical models. In particular, it seems to us that there is no naturally "correct" mathematical entropy jump condition to (1) (or (2)).

Finally, let us point out that the fact that there is a 'crossing' between the left and right flux in our examples is essential. In fact, if there is no flux crossing, as for example in models of traffic flow with variable road surface conditions [3,13], then the two entropy theories outlined herein agree.

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