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A mathematical model of clarifier-thickener units

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We review a recent analysis of a clarifier-thickener model for flocculated suspensions. The resulting governing PDE is a strongly degenerate convection-diffusion equation with discontinuous flux. A new numerical example illustrates how control actions speed up thickener operation significantly.

1 Introduction and mathematical model

Continuously operated clarifier-thickener (CT) units for the solid-liquid separation of suspensions are widely used in engineering applications and wastewater treatment. We here consider a recent extension [3] of the spatially one-dimensional CT model outlined in [2] to suspensions forming compressible sediments. A CT vessel is drawn in Figure 1.

We assume that all variables depend on depth x and time t only. Four zones are distinguished: the thickening zone ($0 < x < x_{\mathrm{R}}$), the clarification zone ($x_{\mathrm{L}} < x < 0$), the underflow zone ($x > x_{\mathrm{R}}$) and the overflow zone ($x < x_{\mathrm{L}}$). The CT is continuously fed at the feed level $x = 0$ with fresh feed suspension of concentration $u_{\mathrm{F}}(t)$ at a volume feed rate $Q_{\mathrm{F}}(t) \geq 0$. The prescribed volume underflow rate, at which the thickened sediment is removed from the unit, is $Q_{\mathrm{R}}(t) \geq 0$. Consequently, the overflow rate is $Q_{\mathrm{L}}(t) = Q_{\mathrm{F}}(t) - Q_{\mathrm{R}}(t)$, where we assume $Q_{\mathrm{L}}(t)$ and $Q_{\mathrm{R}}(t)$ are chosen such that $Q_{\mathrm{F}}(t) \geq 0$. For a vessel with constant cross-sectional area S, we define the velocities $q_{\mathrm{L}}(t) := Q_{\mathrm{L}}(t)/S$ and $q_{\mathrm{R}}(t) := Q_{\mathrm{R}}(t)/S$.

The suspension is characterized by the hindered settling function $b(u)$ and the effective solid stress function $\sigma_{\mathrm{e}}(u)$ [1, 3]. We assume that $b(u)$ is piecewise differentiable with $b(u) = 0$ for $u \leq 0$ or $u \geq u_{\max}$, where $u_{\max}$ is the maximum concentration, $b(u) > 0$ for $0 < u < u_{\max}$, $b'(0) > 0$ and $b'(u_{\max}) \leq 0$. As a typical example we consider $b(u) = v_\infty u(1-u)^C$ for $0 < u < u_{\max}$ and $b(u) = 0$ otherwise, where $C \geq 1$ and $v_\infty > 0$ is the settling velocity of a single floc in pure fluid. The function $\sigma_{\mathrm{e}}(u)$ is assumed to satisfy $\sigma_{\mathrm{e}}(u) \geq 0$ for all u, $\sigma_{\mathrm{e}}'(u) = 0$ and $\sigma_{\mathrm{e}}(u) = 0$ for $u > u_{\mathrm{c}}$. A common formula is the power law $\sigma_{\mathrm{e}}(u) = 0$ for $u \leq u_{\mathrm{c}}$ and $\sigma_{\mathrm{e}}(u) = \sigma_0((u/u_{\mathrm{c}})^k - 1)$ for $u > u_{\mathrm{c}}$, where u_{c} is the critical concentration at which solid particles touch each other, and with parameters $\sigma_0 > 0$ and $k > 1$. Furthermore, we define $A(u) := \int_0^u b(s)\sigma_{\mathrm{e}}'(s)/(\Delta\varrho g s)ds$, where $\Delta\varrho$ is the solid-fluid density difference and g is the acceleration of gravity. We define the vector of discontinuity parameters $\boldsymbol{\gamma} := (\gamma_1, \gamma_2)$ with $\gamma_1(x) = 1$ for $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$, $\gamma_1(x) = 0$ for $x \notin (x_{\mathrm{L}}, x_{\mathrm{R}})$, $\gamma_2(x) = q_{\mathrm{L}}$ for $x < 0$ and $\gamma_2(x) = q_{\mathrm{R}}$ for $x > 0$. Then the complete CT model is given by the strongly degenerate convection-diffusion problem

$$u_t + f(\boldsymbol{\gamma}(x), u)_x = (\gamma_1(x)A(u)_x)_x, \quad x \in \mathbb{R}, \quad t > 0, \qquad f(\boldsymbol{\gamma}(x), v) := \gamma_1(x)b(u) + \gamma_2(x)(u - u_{\mathrm{F}}), \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

2 Numerical scheme and mathematical analysis

The numerical scheme for (1)–(2) is a straightforward extension of the scheme used in [2] for the first-order variant of (1) for ideal suspensions. To define it, choose $\Delta x > 0$ and set $x_j := j\Delta x$, $\boldsymbol{\gamma}_{j+1/2} := \boldsymbol{\gamma}(x_{j+1/2})$ and $U_j^0 := u_0(x_j)$ for $j \in \mathbb{Z}$, where $x_{j+1/2} := x_j + \Delta x/2$. We define the approximations according to the explicit marching formula

$$U_j^{n+1} = U_j^n - \lambda\Delta_- h(\boldsymbol{\gamma}_{j+1/2}, U_{j+1}^n, U_j^n) + (\lambda/\Delta x)\Delta_-(\gamma_{1,j+1/2}\Delta_+ A(U_j^n)), \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $\lambda := \Delta t/\Delta x$, $\Delta_- V_j := V_j - V_{j-1}$, $\Delta_+ V_j := V_{j+1} - V_j$, and $h(\boldsymbol{\gamma}, v, u) := \frac{1}{2}[f(\boldsymbol{\gamma}, u) + f(\boldsymbol{\gamma}, v) - \int_u^v |f_u(\boldsymbol{\gamma}, w)|dw]$ is the Engquist-Osher numerical flux. For easy reference, we denote by u^Δ the piecewise constant function that assumes the value U_j^n on the rectangle $[x_{j-1/2}, x_{j+1/2}) \times [t_n, t_{n+1})$. See [3] for a (more efficient) semi-implicit variant of our scheme and a detailed discussion of its advantages compared with alternative discretizations.

Independently of the smoothness of $\boldsymbol{\gamma}(x)$, solutions to (1) are in general not smooth and weak solutions must be sought. Since discontinuous weak solutions are in general not unique, an entropy condition must be imposed to single out the physically correct solution, the *entropy solution*. When $\boldsymbol{\gamma}$ is smooth, it is well known [5] that there exists a unique and stable entropy solution to (1). The entropy condition and the well-posedness theory for that case are not applicable when $\boldsymbol{\gamma}$ is discontinuous.

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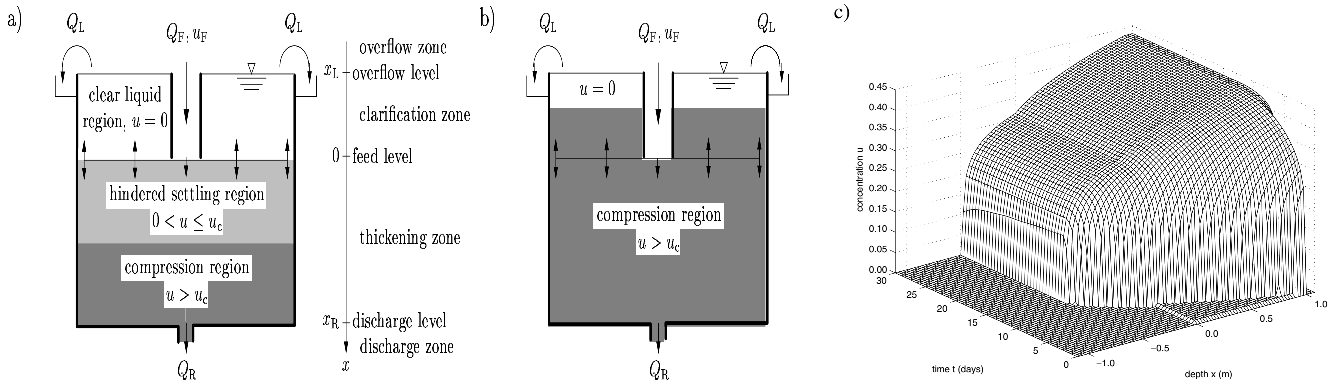

Fig. 1 A clarifier-thickener unit treating a flocculated suspension: (a) steady-state operation in conventional mode, (b) steady-state operation in high-rate mode, (c) simulation of filling up a CT with $\Delta x = 1/400\,\mathrm{m}$, $\lambda = 4000\,\mathrm{s/m}$.

In [3] we suggest a notion of entropy solution that accounts for the discontinuities in $\boldsymbol{\gamma}$. Let $\mathcal{J} := \{x_\mathrm{L}, 0, x_\mathrm{R}\}$ denote the set of discontinuities of $\boldsymbol{\gamma}$. For $m \in \mathcal{J}$, we denote by $\boldsymbol{\gamma}(m-)$ and $\boldsymbol{\gamma}(m+)$ the one-sided limits of $\boldsymbol{\gamma}$ at m. Then we say that a function $u(x,t)$ is a BV_t *entropy solution* of (1)–(2) if it satisfies the following conditions: (1) *Regularity.* $u \in L^1 \cap BV_t$, $u(x,t)$ assumes values in $[0,1]$ only, and $A(u)_x$ is uniformly bounded on $(x_\mathrm{L}, x_\mathrm{R}) \times (0,T)$. (2) *Weak formulation.* For all test functions $\phi(x,t)$, $\int_0^T \int_{\mathbb{R}} (u\phi_t + [f(\boldsymbol{\gamma}(x),u) - \gamma_1(x)A(u)_x]\phi_x)\,dx\,dt = 0$. (3) *Initial condition.* The initial condition is satisfied in the following strong sense: $\lim_{t\downarrow 0} \int_{\mathbb{R}} |u(x,t) - u_0(x)|\,dx = 0$. (4) *Regularity of* $A(u)$. For any $t \in [0,T]$, $x \mapsto A(u(x,t))$ is continuous at $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$. (5) *Entropy condition.* The following entropy inequality holds for all $c \in [0,1]$ and all nonnegative test functions $\phi(x,t)$:

$$\int_0^T \int_{\mathbb{R}} (|u-c|\phi_t + \mathrm{sgn}\,(u-c)\big[f(\boldsymbol{\gamma}(x),u) - f(\boldsymbol{\gamma}(x),c) - \gamma_1(x)A(u)_x\big]\phi_x)\,dx\,dt$$
$$+ \int_0^T \sum_{m\in\mathcal{J}} \big|f(\boldsymbol{\gamma}(m+),c) - f(\boldsymbol{\gamma}(m-),c)\big|\phi(m,t)\,dt \geq 0.$$

Following [4], we proved in [3] that if v and u are two BV_t entropy solutions, then for any $t \in (0,T)$, we have $\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|v(\cdot,0) - u\cdot,0)\|_{L^1(\mathbb{R})}$. The proof of this L^1 stability inequality, which immediately implies uniqueness, relies on jump conditions that relate limits from the right and left of the BV_t entropy solution u at jumps in $\gamma(x)$. In fact, if u is an BV_t entropy solution and $m \in \mathcal{J}$, then the Rankine-Hugoniot condition $[\![f(\boldsymbol{\gamma},u) - \gamma_1 A(u)_x]\!] = 0$ holds for a.e. $t \in (0,T)$, where $[\![\cdot]\!]$ denotes the jump across m. Furthermore, for $[\![u]\!] \neq 0$, the entropy jump condition $[\![F(\boldsymbol{\gamma},u,c) - \mathrm{sgn}\,(u-c)(\gamma_1 A(u)_x)]\!] \leq |[\![f(\boldsymbol{\gamma},c)]\!]|$ for all $c \in [0,1]$ holds, where $F(\boldsymbol{\gamma},u,c) = \mathrm{sgn}\,(u-c)[f(\boldsymbol{\gamma},u) - f(\boldsymbol{\gamma},c)]$.

In [3] we prove (under some technical assumptions on u_0 and supposing that a suitable CFL condition is satisfied) that the scheme converges to a limit u that satisfies all components of our definition of entropy solutions except (4). However, our numerical results support that $A(u)$ is continuous across $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$. In particular, transient numerical simulations converge (for large times) to steady-state solutions.

3 Numerical example

We consider the parameters $v_\infty = 10^{-4}\,\mathrm{m/s}$, $C = 5$, $q_\mathrm{L} = -1.0 \times 10^{-5}\,\mathrm{m/s}$, $q_\mathrm{R} = 2.5 \times 10^{-5}\,\mathrm{m/s}$, $u_\mathrm{F} = 0.086$, $\sigma_0 = 1\,\mathrm{Pa}$, $u_\mathrm{c} = 0.1$, $k = 6$ and $\Delta\varrho = 1500\,\mathrm{kg/m^3}$ corresponding to a hypothetical material. As in Example 3 of [3], we simulate filling up a CT with $x_\mathrm{R} = -x_\mathrm{L} = 1\,\mathrm{m}$ until the high-rate steady-state solution corresponding to these parameters is attained. In [3], the simulated fill-up process takes about a year since the control parameters are not varied. In our example, we keep the feed flux $(q_\mathrm{R} - q_\mathrm{L})Su_\mathrm{F}$ fixed, but vary its division into the thickening and clarification zones by using the piecewise constant control functions $\tilde{q}_\mathrm{R}(t) := \theta(t)q_\mathrm{R}$, $\tilde{q}_\mathrm{L}(t) := (1-\theta(t))q_\mathrm{L}$, where $\theta(t) := (\lfloor t/1\,\mathrm{d}\rfloor/13\,\mathrm{d})^2$ for $t \leq 14\,\mathrm{d}$ and $\theta(t) = 1$ for $t > 14\,\mathrm{d}$. In other words, during two weeks the discharge opening is widened every day until the full value of q_R is reached. We observe that by taking these control actions, the desired steady state is attained about ten times more rapidly than in Example 3 of [3].

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