CONVERGENCE OF A DIFFERENCE SCHEME FOR CONSERVATION LAWS WITH A DISCONTINUOUS FLUX

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Abstract. The subject of this paper is a scalar finite difference algorithm, based on the Godunov or Engquist-Osher flux, for scalar conservation laws where the flux is spatially dependent through a possibly discontinuous coefficient, k. The discretization of k is staggered with respect to the discretization of the conserved quantity u, so that only a scalar Riemann solver is required. The main result of the paper is convergence of a subsequence to a weak solution when the flux is strictly concave and the coefficient k is positive and has bounded variation. The limit solution satisfies a set of entropy inequalities. Satisfaction of these inequalities is shown to rule out nonphysical discontinuities if the limit solution is piecewise smooth.

Key words. conservation laws, difference approximations, discontinuous coefficients

AMS subject classifications. 35L65, 65M06, 65M12, 35R05

1. Introduction.

The subject of this paper is finite difference algorithms for computing approximate solutions of the Cauchy problem for scalar conservation laws of the form

$$u_t + (k(x)f(u))_x = 0, \quad u(x,0) = u_0(x) \tag{1.1}$$

Here the flux $k(x)f(u)$ has a possibly discontinuous spatial dependence through the coefficient $k(x)$, which is allowed to have jump discontinuities. A simple physical model corresponding to 1.1 is the Witham model of car traffic flow on a highway [27]. The spatially varying coefficient $k(x)$ corresponds to changing road conditions, varying speed limits being the most common example. Equations of this type have been addressed by several authors [25], [9], [10], [16] in recent years, often as the simplest examples of nonstrictly hyperbolic systems. Within this framework, 1.1 can be written as a 2x2 system:

$$\begin{aligned} u_t + (kf(u))_x &= 0 \\ k_t &= 0 \end{aligned} \tag{1.2}$$

The two characteristic speeds associated with this system are $\lambda_u = kf'(u)$ and $\lambda_k = 0$. The system fails to be strictly hyperbolic if f' can take both signs, since then the eigenvalues coincide when $f'(u) = 0$.

Even when $k(x)$ and the initial data $u_0(x)$ are smooth, solutions to 1.1 develop discontinuities after finite time, and so weak solutions are sought, which satisfy

$$\int_{t>0}\int_{-\infty}^{+\infty} (\phi_t u + \phi_x kf(u))dxdt + \int_{-\infty}^{+\infty} \phi(x,0)u_0(x)dx = 0 \tag{1.3}$$

for every smooth test function ϕ with compact support in $t \geq 0$, $-\infty < x < +\infty$.

This paper considers the conservation law 1.1 within the framework of scalar finite difference schemes. The spatial domain $\{x : -\infty < x < +\infty\}$ is divided into cells $I_j = [x_j - \Delta x/2, x_j + \Delta x/2)$ with centers at the points $x_j = j\Delta x$ for $j = 0, \pm 1, \pm 2, \ldots$.

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Similarly, the time domain $\{t : t \geq 0\}$ is discretized via $t^n = n\Delta t$ for $n = 0, 1, \ldots$, resulting in time strips $I^n = [t^n, t^{n+1})$. Let $\chi_{j,n}$ be the characteristic function for the rectangle $I_j \times I^n$. The finite difference scheme then generates, for each mesh size $\Delta = \Delta x$, a piecewise constant approximation

$$u^{\Delta}(x,t) = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \chi_{j,n}(x,t) u_j^n \tag{1.4}$$

The initial data, which is assumed to have bounded variation, is discretized via $u_j^0 = (1/\Delta x)\int_{I_j} u_0(x)dx$, resulting in a piecewise constant approximation

$$u^{\Delta}(x,0) = \sum_j \chi_j(x) u_j^0 \tag{1.5}$$

where $\chi_j(x)$ is the characteristic function for the interval I_j. The approximations u_j^n are generated by the explicit time marching algorithm

$$u_j^{n+1} = u_j^n - \lambda(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}} h_{j-\frac{1}{2}}) \tag{1.6}$$

Here, $\lambda = \Delta t/\Delta x$, and $h_{j+\frac{1}{2}}$ is based on a two point monotone (increasing with respect to the second argument, decreasing with respect to the first) numerical flux, $h^f(v,u)$, consistent with the actual flux, i.e., $h^f(u,u) = f(u)$. In order to maintain the monotonicity of the scheme, the arguments of the numerical flux are transposed when the coefficient $k(x)$ is negative, so

$$h_{j+\frac{1}{2}} = \begin{cases} h^f(u_{j+1}, u_j) & \text{if } k_{j+\frac{1}{2}} \geq 0 \\ h^f(u_j, u_{j+1}) & \text{if } k_{j+\frac{1}{2}} < 0 \end{cases}$$

In particular, this paper focuses on the closely-related Godunov and Engquist-Osher fluxes. The Godunov [6] flux is derived by applying the exact solution operator for 1.1, with constant k, to piecewise constant initial data. Dropping the superscript, the numerical flux that results is then $h(v,u) = f(u^G(v,u))$, where $u^G(v,u)$ is the similarity solution of the resulting Riemann problem with right and left states v and u, evaluated anywhere along the vertical half line $t > 0$ in (x,t)-space where the jump in the initial data occurs. The following formula for the Godunov flux is derived in [18]:

$$h(v,u) = \begin{cases} \min_{[u,v]} f(w) & \text{if } u \leq v \\ \max_{[v,u]} f(w) & \text{if } u \geq v \end{cases} \tag{1.8}$$

The Godunov flux is Lipschitz continuous, with piecewise continuous partial derivatives. With the notational convention that $a_- = \min(a,0)$, $a_+ = \max(a,0)$, it follows from 1.8 that

$$f'_-(v) \leq h_v(v,u) \leq 0 \leq h_u(v,u) \leq f'_+(u) \tag{1.9}$$

The Engquist-Osher (EO) flux can be derived by discretizing the transport collapse operator, in a manner analogous to the derivation of the Godunov flux from the exact solution operator [1]. It is defined [4] by:

$$h(v,u) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}\int_u^v |f'(w)|dw \tag{1.10}$$

The EO flux is also Lipschitz continuous, but is smoother than the Godunov flux. It has continuous partial derivatives satisfying

$$f'_-(v) = h_v(v,u) \le 0 \le h_u(v,u) = f'_+(u) \tag{1.11}$$

It follows from 1.9 and 1.11 that $\|f'\|_\infty$ serves as a Lipschitz constant for both fluxes. In the case where the flux f is convex or concave, with a single stagnation point, the EO and Godunov fluxes are identical, except for the single case where the two arguments are joined by a sonic shock, i.e., $f'(v) < 0 < f'(u)$. In addition, both fluxes satisfy $h^{kf} = kh^f$ for $k \ge 0$, which greatly simplifies the analysis that follows. An example of a flux which does not have this property is the Lax-Friedrichs flux (see Section 5).

In the scheme 1.6, the coefficient $k(x)$, which is assumed to have bounded variation, is approximated at each cell boundary, resulting in a discretized version of k,

$$k^\Delta(x) = \sum_j \chi_{j+\frac{1}{2}}(x) k_{j+\frac{1}{2}} \tag{1.12}$$

where $\chi_{j+\frac{1}{2}}(x)$ is the characteristic function for the interval $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$. The main requirement for the method of generating the approximations $k_{j+\frac{1}{2}}$ is consistency, i.e.,

$$k^\Delta \to k \quad \text{in } L^1 \text{ as } \Delta \to 0 \tag{1.13}$$

The centered approximation

$$k_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k(x)dx \tag{1.14}$$

satisfies 1.13, but the approximation need not be centered. For example, since k has bounded variation, either of the biased approximations

$$k_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} k(x)dx \quad \text{or} \quad k_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{3}{2}}} k(x)dx \tag{1.15}$$

also satisfy 1.13, and may be more convenient in some situations. The discretized version k^Δ has jumps at cell centers, as opposed to cell boundaries. In other words, the discretization of k is staggered with respect to the discretization of u. This results in a significant reduction in complexity, both computationally, and analytically, when compared to the approach where the two discretizations are aligned. The concept of reducing complexity via mesh staggering is well known. For example, the author of [24] points out that the Lax-Friedrichs scheme (for constant k), which requires no Riemann solvers, can be viewed as the Godunov scheme, applied over a staggered mesh. In fact, given the simplicity of the scheme presented here, it is likely that the author of this paper is not the first to have derived or implemented it. Nevertheless, there does not appear to be any published convergence analysis for scalar schemes when k is discontinuous.

When the discretizations of the conserved quantity and flux coefficient are aligned, more complicated 2x2 Riemann problems arise, and thus a 2x2 Riemann solver is required. In [25], Temple used the Glimm scheme [5], solving 2x2 Riemann problems,

to establish existence and uniqueness for a somewhat more general system of resonant conservation laws than 1.2. In [16] and [17], the authors studied the 2x2 Godunov method as it applies to those conservation laws. They established that the 2x2 Godunov method is convergent (modulo extraction of a subsequence) to a weak solution. As in this paper, the singular function Ψ was used to establish compactness. They established a bound on the total variation of a function that corresponds to the function z^{Δ} of this paper, but their method of analysis was different, entailing a study of the various waves arising from the 2x2 Riemann problems. They also showed that, with the 2x2 Godunov method, the total variation of the conserved quantity u^{Δ} can grow at most linearly, with some mild assumptions about the coefficient k. In [17] it was observed that time independent bounds on derivatives (measured via the singular function Ψ) had been achieved for the 2x2 Glimm and Godunov schemes, but not for any scalar scheme that applies to

$$u_t + f(k(x), u)_x = 0 \tag{1.16}$$

Section 3 of this paper provides a bound of this type for equations of the form 1.1, which are an important subset of 1.16.

Outside of the difference scheme methods, the front tracking approach has been very effective, especially as an analytical tool for studying the conservation law 1.1. In particular, [10] establishes existence, uniqueness, and asymptotic behavior of the solution of 1.1, under essentially the same concavity assumptions about the flux f as this paper. The singular function Ψ is used in [10] also. A bound on the variation, measured via Ψ, is achieved by studying the wave interactions that arise. Additionally, satisfaction of a wave entropy condition is established for the limit of the front tracking scheme, and uniqueness is shown to follow from this entropy condition.

Section 2 of this paper discusses the monotonicity and L^1-contractiveness properties of the scheme. Section 3 establishes the main result of the paper, convergence of a subsequence to a weak solution of the conservation law. Section 4 provides a verification that the limit function satisfies geometric entropy conditions with the additional assumption that k is piecewise smooth. Section 5 contains a short discussion of the Lax-Friedrichs version of the scheme. Section 6 briefly discusses more general spatial dependence of the flux, and Section 7 presents several examples of applications of the numerical scheme. An appendix enumerates the solutions to the Riemann problem for 1.1, and proves some elementary properties needed for Section 4.

2. Monotonicity. For the case of constant k, the theory of monotone schemes has been essentially complete for many years [8], [2] ,[12], [22]. In that setting, approximations generated by monotone schemes are well-known to share many of the properties of the actual weak solution to the conservation law 1.1, including monotonicity, L^1-contractiveness, and satisfaction of a discrete entropy inequality. Their major drawback is that they are at best only first order accurate even in regions where the solution is smooth [8]. Nevertheless, they provide the starting point for many of the modern higher order accurate schemes, which are constructed by modifying the two-point monotone flux function with higher order correction terms . Then flux limiters [26], [19], [7] are applied to damp out the spurious oscillations that these higher order corrective terms often give rise to in regions of rapid transition.

Denote the time advance operator for the scheme 1.6 by Γ, so that $\Gamma(U^n) = U^{n+1}$, with $U^n = (\dots u^n_{-2}, u^n_{-1}, u^n_0, u^n_1, u^n_2 \dots)$ denoting the approximate solution at time level t^n. The scheme is order-preserving, or monotone if

$$U^n \le V^n \Rightarrow \Gamma(U^n) \le \Gamma(V^n) \tag{2.1}$$

When k is constant, monotonicity of the scheme follows from monotonicity of the numerical flux h under suitable CFL conditions. It follows that, in addition, the computed approximations U^n remain within the convex hull of the initial data for all n. The following proposition provides an analog of these properties for the variable k situation.

PROPOSITION 2.1. *With the CFL condition $2\lambda\|k\|_\infty\|f'\|_\infty \leq 1$ both the Godunov and EO versions of the scheme are monotone. If $k(x)$ is nonnegative or nonpositive the CFL condition can be relaxed to $\lambda\|k\|_\infty\|f'\|_\infty \leq 1$. If the initial data $u_0(x)$ lies within the interval $\Omega_0 = [u_{min}, u_{max}]$, and $f(u_{min}) = f(u_{max}) = 0$, then the computed solution will remain within Ω_0.*

Proof. Let $u_j = u_j^n$; no confusion will arise since only two successive time levels are of interest. Expressing the three-point scheme as $u_j^{n+1} = G(u_{j-1}, u_j, u_{j+1})$, the goal is to show that the partial derivatives $\partial G/\partial u_{j+i}$ are each nonnegative, from which monotonicity of the whole scheme Γ follows. That the partial derivatives of G with respect to u_{j-1} and u_{j+1} are nonnegative is seen from the following formulas.

$$\partial G/\partial u_{j-1} = \begin{cases} \lambda k_{j-\frac{1}{2}} h_u(u_j, u_{j-1}) & \text{if } k_{j-\frac{1}{2}} \geq 0 \\ \lambda k_{j-\frac{1}{2}} h_v(u_{j-1}, u_j) & \text{if } k_{j-\frac{1}{2}} \leq 0 \end{cases} \tag{2.2}$$

$$\partial G/\partial u_{j+1} = \begin{cases} -\lambda k_{j+\frac{1}{2}} h_v(u_{j+1}, u_j) & \text{if } k_{j+\frac{1}{2}} \geq 0 \\ -\lambda k_{j+\frac{1}{2}} h_u(u_j, u_{j+1}) & \text{if } k_{j+\frac{1}{2}} \leq 0 \end{cases} \tag{2.3}$$

For $\partial G/\partial u_j$, four cases have to be considered, depending on the signs of $k_{j-\frac{1}{2}}$ and $k_{j+\frac{1}{2}}$. If $k_{j-\frac{1}{2}} \geq 0$ and $k_{j+\frac{1}{2}} \geq 0$,

$$\begin{aligned} \frac{\partial G}{\partial u_j} &= 1 - \lambda k_{j+\frac{1}{2}} h_u(u_{j+1}, u_j) + \lambda k_{j-\frac{1}{2}} h_v(u_j, u_{j-1}) \\ &\geq 1 - \lambda k_{j+\frac{1}{2}} f'_+(u_j) + \lambda k_{j-\frac{1}{2}} f'_-(u_j) \\ &\geq 1 - \lambda\|k\|_\infty f'_+(u_j) + \lambda\|k\|_\infty f'_-(u_j) \\ &= 1 - \lambda\|k\|_\infty |f'(u_j)| \geq 1 - \lambda\|k\|_\infty\|f'\|_\infty \geq 0 \end{aligned}$$

The case where $k_{j-\frac{1}{2}} \leq 0$ and $k_{j+\frac{1}{2}} \leq 0$ is similar, and the calculation is omitted. For the case where $k_{j-\frac{1}{2}} < 0$ and $k_{j+\frac{1}{2}} > 0$,

$$\begin{aligned} \frac{\partial G}{\partial u_j} &= 1 - \lambda k_{j+\frac{1}{2}} h_u(u_{j+1}, u_j) + \lambda k_{j-\frac{1}{2}} h_u(u_{j-1}, u_j) \\ &\geq 1 - \lambda k_{j+\frac{1}{2}} f'_+(u_j) + \lambda k_{j-\frac{1}{2}} f'_+(u_j) \\ &\geq 1 - 2\lambda\|k\|_\infty f'_+(u_j) \geq 1 - 2\lambda\|k\|_\infty\|f'\|_\infty \geq 0 \end{aligned}$$

The case where $k_{j-\frac{1}{2}} > 0$ and $k_{j+\frac{1}{2}} < 0$ is similar and is omitted. The stated invariance of Ω_0 with respect to the computed solution u_j^n follows from the monotonicity of the scheme, along with the fact that f vanishes at u_{min} and u_{max}. Specifically,

$$\begin{aligned} u_{min} = G(u_{min}, u_{min}, u_{min}) &\leq G(u_{j-1}, u_j, u_{j+1}) \\ &\leq G(u_{max}, u_{max}, u_{max}) = u_{max} \end{aligned}$$

□

The condition $f(u_{min}) = f(u_{max}) = 0$, required for invariance of the initial domain Ω_0, is not required in the constant k setting. This difference arises from the fact that, for constant k, $G(u,u,u) = u$; this is not generally true when k is variable. This difference is related to the fact that solutions of 1.1 with constant initial data, but variable k, do not generally remain constant. (See the example in Section 7.3.)

For the remainder of this paper it will be assumed that $u_0 \in L^1 \bigcap L^\infty \bigcap BV$ and $k \in L^\infty \bigcap BV$. A locally integrable function $w : \mathbf{R^1} \to \mathbf{R^1}$ is in BV, the space of functions having bounded variation if

$$\sup_{|h|>0} \int_{-\infty}^{+\infty} \frac{|w(x+h) - w(x)|}{|h|} dx < +\infty \tag{2.4}$$

The quantity on the left of 2.4 is the total variation of w, denoted $TV(w)$. Also, it will be assumed that k is constant for large x, specifically, that there are constants $k(+\infty)$, $k(-\infty)$, and X such that $k(x) = k(-\infty)$ for $x \le -X$, $k(x) = k(+\infty)$ for $x \ge +X$. With the cell-average type discretizations discussed in Section 1 for u_0 and k, the discretized versions also satisfy these assumptions, and the various norms of the discretized quantities are bounded uniformly in Δ by the corresponding norms for u_0 and k. For example,

$$TV(k^\Delta) \le TV(k), \;\; TV(u^\Delta(\cdot,0)) \le TV(u_0) \tag{2.5}$$

The following proposition is the main reason for establishing the monotonicity of the scheme.

Proposition 2.2. *With the CFL conditions in Proposition 2.1 and the assumption that the initial data satisfies u_0, $v_0 \in L^1 \bigcap L^\infty$, and $k \in L^\infty$ and is constant for large x, the scheme 1.6 is L_1-contractive, i.e., the inequality*

$$\sum_j |v_j^{n+1} - u_j^{n+1}| \Delta x \le \sum_j |v_j^n - u_j^n| \Delta x \tag{2.6}$$

holds for the pair of computed approximated solutions $\{u_j^n\}$ and $\{v_j^n\}$ generated by the scheme. The following inequality also holds:

$$\sum_j |u_j^{n+1} - u_j^n| \Delta x \le \sum_j |u_j^1 - u_j^0| \Delta x \tag{2.7}$$

Proof. Both solutions $\{u_j^n\}$ and $\{v_j^n\}$ remain in $L^1 \bigcap L^\infty$. The L^∞ bound follows from Proposition 2.1. The L^1 bound follows from the finite range of influence of the initial data, along with the assumption that k is constant for large x. These observations along with the fact that the time advance operator Γ is monotone (due to the CFL condition) and conservative, add up to the hypotheses of the Crandall-Tartar Lemma [2], [3], giving 2.6. The inequality 2.7 follows from L_1-contractiveness, applied inductively, to two successive time steps, U^n and U^{n+1}, of a single computed solution. □

3. Stability and convergence. For scalar conservation laws, total variation stability is an important concept, both theoretically, and practically. From a theoretical point of view, algorithms which generate solutions having bounded total variation are convergent in the sense that they generate convergent subsequences as the mesh

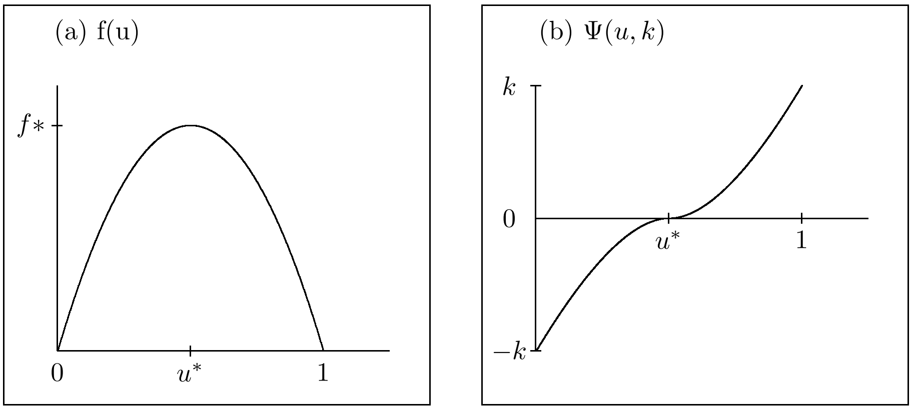

FIG. 3.1. *The concave flux $f(u)$ and singular function $\Psi(u, k)$.*

size tends to zero. If, in addition, a discrete entropy condition is satisfied, the whole computed sequence converges to the physically correct solution. On the practical side, a bound on the total variation is an indication of limited, or possibly zero, spurious oscillations in the computed solution. In the case where the coefficent $k(x)$ is constant, L_1 contractiveness implies, via shift-invariance of the numerical solution operator, that the scheme is Total Variation Decreasing (TVD). However, even for smooth, but nonconstant, $k(x)$, the total variation of the solution of the conservation law generally increases, at least initially, and so there is no hope that the numerical scheme for variable k will be TVD.

As in [25], [9], [10], [16], the approach to stability in this paper is to use the singular mapping Ψ first used by Temple [25] in his study of resonant hyperbolic systems. Some assumptions about the flux function $f(u)$ will be required before defining Ψ. The flux function $f \in C^2[0,1]$ is assumed to be strictly concave on $[0,1]$, i.e., $f''(u) \le \mu < 0$, and $f(0) = f(1) = 0$. Within the interval $(0,1)$, $f(u) > 0$, with a single maximum f^* at $u^* \in (0,1)$. The simplest example is $f(u) = u(1-u)$, in which case $u^* = 1/2$, $f^* = 1/4$. See Figure 3.1.

Then the singular function Ψ is defined by

$$\Psi(u,k) = k\sigma(u-u^*)(1 - f(u)/f^*) = \frac{k}{f^*}\int_{u^*}^{u} |f'(w)|dw \tag{3.1}$$

Here $\sigma(w) = w/|w|$ if $w \neq 0$, and $\sigma(0) = 0$. In the remainder of this paper, unless explicitly stated otherwise, $k(x)$ will be assumed to be bounded and strictly positive: $0 < \underline{k} \le k(x) \le \overline{k}$. Then 1.7 becomes $h_{j+\frac{1}{2}} = h(u_{j+1}, u_j)$. The convergence result of this paper remains valid (with a more restrictive CFL condition, and a larger bound on the variation of z^Δ) if $k(x)$ is allowed to have finitely many points x where $k(x^-)k(x^+) \le 0$. In order to avoid obscuring the main idea of the paper, the analysis of this more general case is not presented.

For each value of k in $[\underline{k}, \overline{k}]$, $\Psi(\cdot, k) : [0,1] \to [-k, k]$ is an increasing, $1-1$ mapping. It is regular everywhere, except at the stagnation point u^*, where both f' and $\partial\Psi/\partial u$ vanish. See Figure 3.1. The following Lipschitz continuity relationships in u and k follow directly from the definition of Ψ and the assumptions about the flux f.

$$|\Psi(u,k_1) - \Psi(u,k_2)| \le |k_1 - k_2| \tag{3.2}$$

$$|\Psi(u_1,k)-\Psi(u_2,k)|\le k\frac{\|f'\|_\infty}{f^*}|u_1-u_2| \tag{3.3}$$

The function Ψ maps a function $w(x,t)$ into a new function $z(x,t)$ via $z(x,t) = \Psi(w(x,t),k(x))$. The following elementary facts about z are readily verified.

PROPOSITION 3.1. *Let $w(x,t) \in [0,1]$, and $k(x) \in [\underline{k},\overline{k}]$.*

1. For each $t \ge 0$, $z(\cdot,t) \in L^\infty$, and $\|z(\cdot,t)\|_\infty \le \|k\|_\infty = \overline{k}$.

2. For each $t \ge 0$, $z(\cdot,t) \in L^1_{loc}$, and $\int_{-B}^{+B} |z(x,t)|dx \le 2B\overline{k}$.

It is now possible to state the main result of this paper.

THEOREM 3.2. *Let $k \in BV \bigcap L^\infty$, with k taking constant values for large x, and assume that $0 < \underline{k} \le k(x) \le \overline{k}$. Let $f \in C^2[0,1]$, $f'' \le \mu < 0$, $f > 0$ in $(0,1)$, with a single maximum $f^* = f(u^*)$ at $u^* \in (0,1)$. Let $u_0 \in BV \bigcap L^1 \bigcap L^\infty$. Let the mesh size $\Delta \to 0$ with λ fixed and satisfying the CFL condition $\lambda\|k\|_\infty\|f'\|_\infty \le 1$, resulting in the sequence of approximations u^Δ. Then, there is a weak solution u of 1.3 and a subsequence u^{Δ_i} such that $u^{\Delta_i} \to u$ a.e. and in L^1_{loc}.*

The rest of this section carries out the analysis required to prove Theorem 3.2. The basic approach is standard as in [22], with the exception that the quantity of immediate interest is $z^\Delta = \Psi(u^\Delta, k^\Delta)$ instead of the computed approximation u^Δ. Compactness for z^Δ follows in the usual way from bounds on the total variation, along with the $L^1_{loc}[-B,B]$ and L^∞ norms, and a time-continuity estimate. Once the existence of a subsequential limit z has been established, the invertibility of Ψ then allows the corresponding weak solution u to be recovered from the limit z, with $u^\Delta \to u$ guaranteed by the continuity of Ψ. Finally, a version of the Lax-Wendroff Theorem [14], modified for the setting of this paper, guarantees that the limit function u is a weak solution of 1.1 .

The following lemma will be required in the proof of the stability theorem that follows.

LEMMA 3.3. *Let $\phi(u) = \sigma(u-u^*)(f^* - f(u)) = f^*\Psi(u,k)/k$ and $\phi_j = \phi(u_j)$. For both the Godunov and EO versions of the scheme,*

$$(\phi_j - \phi_{j+1})_+ \le \chi_-(u_j)|h_{j+\frac12} - h_{j-\frac12}| + \chi_+(u_{j+1})|h_{j+\frac32} - h_{j+\frac12}| \tag{3.4}$$

where $\chi_+(u) = 1$ if $f'(u) > 0$, and 0 elsewhere; similarly, $\chi_-(u) = 1$ if $f'(u) < 0$ and 0 elsewhere.

Proof. Like $\Psi(u,k)$, ϕ is a strictly increasing function of u, so $(\phi_j - \phi_{j+1})_+ > 0$ iff $u_j > u_{j+1}$. So, assume that $u_j > u_{j+1}$. The proof will make use of the fact that for concave f, and $u_j \ge u_{j+1}$, the Godunov and EO fluxes are identical, so 1.11 holds for both fluxes. Also, for either flux, the following fact is easily verified

$$h_{j+\frac12} - h_{j-\frac12} = \int_{u_{j-1}}^{u_j} h_u(u_j,w)dw + \int_{u_j}^{u_{j+1}} h_v(w,u_j)dw \tag{3.5}$$

Using the following decomposition of $\phi_j - \phi_{j+1}$,

$$\begin{aligned}\phi_j - \phi_{j+1} &= \int_{u_{j+1}}^{u_j} \phi'(w)dw = \int_{u_{j+1}}^{u_j} |f'(w)|dw \\ &= \int_{u_{j+1}}^{u_j} f'_+(w)dw - \int_{u_{j+1}}^{u_j} f'_-(w)dw\end{aligned}$$

(3.6)

it suffices to show that

$$\chi_-(u_j)|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| \geq \int_{u_j}^{u_{j+1}} f'_-(w)dw \tag{3.7}$$

and

$$\chi_+(u_{j+1})|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}| \geq -\int_{u_j}^{u_{j+1}} f'_+(w)dw \tag{3.8}$$

For 3.7, if $\chi_-(u_j) = 0$ then both u_j and u_{j+1} are to the left of u^*, so the integral vanishes, and the inequality holds. So, assume that $\chi_-(u_j) = 1$, i.e., $u_j > u^*$. Since $u_{j+1} \leq u_j$, $\int_{u_j}^{u_{j+1}} h_v(w, u_j)dw \geq 0$. If $u_{j-1} \geq u^*$, $\int_{u_{j-1}}^{u_j} h_u(u_j, w)dw = 0$, while if $u_{j-1} < u^*$, $\int_{u_{j-1}}^{u_j} h_u(u_j, w)dw \geq 0$. In either case, 3.7 holds, since then

$$\begin{aligned}|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| &= \int_{u_{j-1}}^{u_j} h_u(u_j, w)dw + \int_{u_j}^{u_{j+1}} h_v(w, u_j)dw \\ &\geq \int_{u_j}^{u_{j+1}} h_v(w, u_j)dw = \int_{u_j}^{u_{j+1}} f'_-(w)dw\end{aligned}$$

For 3.8, if $\chi_+(u_{j+1}) = 0$ then both u_j and u_{j+1} are to the right of u^*, so the integral vanishes, and the inequality holds. So, assume that $\chi_+(u_{j+1}) = 1$, i.e., $u_{j+1} < u^*$. Since $u_{j+1} \leq u_j$, $\int_{u_j}^{u_{j+1}} h_u(u_{j+1}, w)dw \leq 0$. If $u_{j+2} \leq u^*$, $\int_{u_{j+1}}^{u_{j+2}} h_v(w, u_{j+1})dw = 0$, while if $u_{j+2} > u^*$, $\int_{u_{j+1}}^{u_{j+2}} h_v(w, u_{j+1})dw \leq 0$. In either case, 3.8 holds, since

$$\begin{aligned}|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}| &= -\int_{u_j}^{u_{j+1}} h_u(u_{j+1}, w)dw - \int_{u_{j+1}}^{u_{j+2}} h_v(w, u_{j+1})dw \\ &\geq -\int_{u_j}^{u_{j+1}} h_u(u_{j+1}, w)dw = -\int_{u_j}^{u_{j+1}} f'_+(w)dw\end{aligned}$$

□

The following theorem plays the role that a uniform bound on the total variation of the computed solution usually plays in the constant k setting. The symbols Δ_+ and Δ_- will be used to denote forward and backward difference operators. For example, $\Delta_+ h_{j-\frac{1}{2}} = \Delta_- h_{j+\frac{1}{2}} = h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$.

Theorem 3.4. *Let $z^n(x) = \Psi(u^\Delta(x, t^n), k^\Delta(x))$. With the assumptions stated in Theorem 3.2, the total variation of z^n is bounded for all $n \geq 0$. Specifically,*

$$TV(z^n) \leq k(-\infty) - k(+\infty) + 6TV(k) + \frac{4\overline{k}}{f^*}\|f'\|_\infty TV(U^0) \tag{3.9}$$

Proof. The superscript n is suppressed in the proof, except where two time levels are of interest. Taking into account the jumps in z at the cell boundaries (due to jumps in u^Δ) and the jumps at the cell centers (due to jumps in k^Δ), the total variation of $z(x)$ is

$$TV(z) = \sum_j |\Delta_+^u z_j| + \sum_j |\Delta_+^k z_{j-\frac{1}{2}}| \tag{3.10}$$

where $\Delta_+^u z_j = \Psi(u_{j+1}, k_{j+\frac{1}{2}}) - \Psi(u_j, k_{j+\frac{1}{2}})$ and $\Delta_+^k z_{j-\frac{1}{2}} = \Psi(u_j, k_{j+\frac{1}{2}}) - \Psi(u_j, k_{j-\frac{1}{2}})$. The second sum in 3.10, due to jumps in k^Δ, is bounded by $TV(k)$, using 3.2. For the first sum, summing over all of the jumps, and using the fact that $z(x) \to -k(\pm\infty)$ as $x \to \pm\infty$, results in

$$\sum_j \Delta_+^u z_j + \sum_j \Delta_+^k z_{j-\frac{1}{2}} = k(-\infty) - k(+\infty) \tag{3.11}$$

It follows from 3.11 that

$$\begin{aligned}
\sum_j (\Delta_+^u z_j)_+ &= k(-\infty) - k(+\infty) - \sum_j (\Delta_+^u z_j)_- - \sum_j \Delta_+^k z_{j-\frac{1}{2}} \\
&\le k(-\infty) - k(+\infty) - \sum_j (\Delta_+^u z_j)_- + \sum_j |\Delta_+^k z_{j-\frac{1}{2}}| \\
&\le k(-\infty) - k(+\infty) - \sum_j (\Delta_+^u z_j)_- + TV(k)
\end{aligned} \tag{3.12}$$

Then

$$\begin{aligned}
\sum_j |\Delta_+^u z_j| &= \sum_j (\Delta_+^u z_j)_+ - \sum_j (\Delta_+^u z_j)_- \\
&\le k(-\infty) - k(+\infty) - 2\sum_j (\Delta_+^u z_j)_- + TV(k)
\end{aligned} \tag{3.13}$$

The rest of the proof amounts to estimating the term $-\sum_j (\Delta_+^u z_j)_-$.

$$\begin{aligned}
-\sum_j (\Delta_+^u z_j)_- &= \sum_j (\Psi(u_j, k_{j+\frac{1}{2}}) - \Psi(u_{j+1}, k_{j+\frac{1}{2}}))_+ \\
&= \frac{1}{f^*} \sum_j k_{j+\frac{1}{2}} (\phi_j - \phi_{j+1})_+
\end{aligned} \tag{3.14}$$

By Lemma 3.3,

$$\begin{aligned}
k_{j+\frac{1}{2}}(\phi_j - \phi_{j+1})_+ &\le k_{j+\frac{1}{2}}(\chi_-(u_j)|\Delta_+ h_{j-\frac{1}{2}}| + \chi_+(u_{j+1})|\Delta_+ h_{j+\frac{1}{2}}|) \\
&\le \chi_-(u_j)\big(|\Delta_+(k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})| + |h_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}}|\big) \\
&+ \chi_+(u_{j+1})\big(|\Delta_+(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}})| + |h_{j+\frac{3}{2}} \Delta_+ k_{j+\frac{1}{2}}|\big) \\
&\le \chi_-(u_j)\big(|\Delta_+(k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})| + f^*|\Delta_+ k_{j-\frac{1}{2}}|\big) \\
&+ \chi_+(u_{j+1})\big(|\Delta_+(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}})| + f^*|\Delta_+ k_{j+\frac{1}{2}}|\big)
\end{aligned} \tag{3.15}$$

The last inequality uses the fact that $|h_{j+\frac{1}{2}}/f^*| \le 1$, which is easily verified for both the Godunov and EO flux. Substituting this into 3.14, summing over j, and shifting indices, gives

$$-\sum_j (\Delta_+^u z_j)_- \le \frac{1}{f^*} \sum_j ((\chi_+(u_j) + \chi_-(u_j))|\Delta_+(k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})| + TV(k)$$

$$\le \frac{1}{\lambda f^*}\sum_j |u_j^{n+1}-u_j^n| + TV(k)$$
$$\le \frac{1}{\lambda f^*}\sum_j |u_j^{1}-u_j^0| + TV(k) \tag{3.16}$$

The second inequality uses the definition of the numerical scheme 1.6. The last inequality uses the L^1-contractiveness of the scheme. The final step in the proof is to estimate the term $\sum_j |u_j^1 - u_j^0|$. For the jth term in this sum,

$$\begin{aligned}|u_j^1 - u_j^0| &\le \lambda k_{j+\frac{1}{2}} |h(u_{j+1}^0, u_j^0) - h(u_j^0, u_{j-1}^0)| + \lambda h(u_j^0, u_{j-1}^0)|\Delta_+ k_{j-\frac{1}{2}}| \\ &\le \lambda k_{j+\frac{1}{2}} \|f'\|_\infty (|u_{j+1}^0 - u_j^0| + |u_j^0 - u_{j-1}^0|) + \lambda f^* |\Delta_+ k_{j-\frac{1}{2}}|\end{aligned} \tag{3.17}$$

The second inequality is due to the fact that $\|f'\|_\infty$ serves as a Lipschitz constant for both the EO and Godunov flux. Summing over j gives

$$\sum_j |u_j^1 - u_j^0| \le \; 2\lambda \overline{k} \|f'\|_\infty TV(U^0) + \lambda f^* TV(k) \tag{3.18}$$

which provides the desired bound on $\sum_j |u_j^1 - u_j^0|$. Substituting this into the last inequality in 3.16 gives

$$-\sum_j (\Delta_+^u z_j)_- \le \frac{2\overline{k}}{f^*}\|f'\|_\infty TV(U^0) + 2TV(k) \tag{3.19}$$

When all of the terms are collected, the bound 3.9 stated in the theorem results. ∎

In addition to the spatial stability estimate provided by Theorem 3.4, the following time continuity estimate for z^Δ follows from L^1-contractiveness.

Lemma 3.5. *With the assumptions stated in Theorem 3.2, there is a constant L, independent of the mesh size Δ, such that for $n > m \ge 0$*

$$\int_{-\infty}^{+\infty} |z^\Delta(x,t^n) - z^\Delta(x,t^m)|dx \le L(n-m)\Delta t \tag{3.20}$$

Proof. Taking into account the constant values of z^Δ in each half-cell, the integral on the right side of 3.20 is

$$\frac{\Delta x}{2}\sum_j |\Psi(u_j^n, k_{j-\frac{1}{2}}) - \Psi(u_j^m, k_{j-\frac{1}{2}})| + \frac{\Delta x}{2}\sum_j |\Psi(u_j^n, k_{j+\frac{1}{2}}) - \Psi(u_j^m, k_{j+\frac{1}{2}})|$$

An application of 3.3 gives

$$\begin{aligned}\int_{-\infty}^{+\infty} |z^\Delta(x,t^n) - z^\Delta(x,t^m)|dx &\le \Delta x \overline{k}\frac{\|f'\|_\infty}{f^*}\sum_j |u_j^n - u_j^m| \\ &\le \Delta x \overline{k}\frac{\|f'\|_\infty}{f^*}\sum_{\nu=m}^{n-1}\sum_j |u_j^{\nu+1} - u_j^\nu| \\ &\le \frac{\Delta t}{\lambda}\overline{k}\frac{\|f'\|_\infty}{f^*}(n-m)\sum_j |u_j^1 - u_j^0|\end{aligned} \tag{3.21}$$

The third inequality above is due to L^1-contractiveness. The proof is completed by bounding the term $\sum_j |u_j^1 - u_j^0|$ independently of Δ, using 3.18 in the proof of Theorem 3.4. $\square$

In the constant $k(x)$ setting, where the scheme 1.6 generates u^Δ with uniformly bounded total variation, the classical theorem of Lax and Wendroff [14] can be invoked to guarantee that if a subsequence u^Δ generated by the finite difference scheme converges boundedly almost everywhere to some function u, then u is a weak solution of the conservation law [15]. The content of the next theorem is that the conclusion of the Lax-Wendroff theorem holds in the variable k setting also. However, with no bound on $TV(u^\Delta)$ available, the proof must be modified. It turns out that the uniform bound on $TV(z^\Delta)$ established in Theorem 3.4 can be used in place of $TV(u^\Delta)$.

THEOREM 3.6. *Let u^Δ be a sequence of approximations computed via the scheme 1.6 which converges in $L^1_{loc}(t > 0, -\infty < x < +\infty)$, and with $TV(z^n)$ uniformly bounded, as results from Theorem 3.4, to a function u. Then u is a weak solution of 1.1, i.e., it satisfies 1.3.*

Proof. Let ψ be a test function with compact support in $t \geq 0, -\infty < x < +\infty$ and $\Delta^t \psi_j^n = \psi_j^{n+1} - \psi_j^n$, $\Delta^x \psi_j^n = \psi_{j+1}^n - \psi_j^n$. Fix the time $0 \leq T = N\Delta t$ such that $\psi = 0$ for $t \geq T$. As in [14], multiply the scheme 1.6 by $\psi(x_j, t^n) = \psi_j^n$, sum over all j and $0 \leq n \leq T/\Delta t$, then sum by parts to get

$$\Delta t \sum_j \sum_n (u_j^n(\Delta^t \psi_j^n/\Delta t) + k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} (\Delta^x \psi_j^n / \Delta x)) + \sum_j \psi_j^0 u_j^0 = 0 \tag{3.22}$$

where $h_{j+\frac{1}{2}} = h(u^\Delta(x_{j+1}, t^n), u^\Delta(x_j, t^n))$. By standard arguments [15], the sums involving $u_j^n(\Delta^t \psi_j^n/\Delta t)$ and $\psi_j^0 u_j^0$ converge to their integral counterparts in 1.3, as $\Delta \to 0$. It remains to show that

$$\Delta x \Delta t \sum_j \sum_n k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} (\Delta^x \psi_j^n / \Delta x) \to \int_{t>0} \int_{-\infty}^{+\infty} k f(u) \psi_x dx dt \tag{3.23}$$

The proof of 3.23 reduces to verifying that

$$\Delta t \Delta x \sum_{n=0}^{N} \sum_j |k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k(x_j) f(u(x_j, t^n))| \to 0 \tag{3.24}$$

For 3.24, an estimate of $k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k(x_j) f(u(x_j, t^n))$ is requred:

$$\begin{aligned} |k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k(x_j) f(u(x_j, t^n))| \leq{}& k_{j+\frac{1}{2}} |h_{j+\frac{1}{2}} - f(u^\Delta(x_j, t^n))| \\ &+ k_{j+\frac{1}{2}} |f(u^\Delta(x_j, t^n)) - f(u(x_j, t^n))| \\ &+ f(u(x_j, t^n)) |k(x_j) - k_{j+\frac{1}{2}}| \end{aligned} \tag{3.25}$$

Due to the smoothness of f, contributions to 3.24 arising from the second term on the right of 3.25 tend to zero by the bounded convergence theorem. For the third term, convergence to zero follows from the bounded convergence theorem again, and the fact that $k^\Delta(x) \to k(x)$ in L^1. The first term on the right of 3.25 remains to be estimated:

$$k_{j+\frac{1}{2}} |h_{j+\frac{1}{2}} - f(u^\Delta(x_j, t^n))| = k_{j+\frac{1}{2}} |h(u_{j+1}^n, u_j^n) - h(u_j^n, u_j^n)|$$

$$= k_{j+\frac{1}{2}}\Big|\int_{u_j^n}^{u_{j+1}^n} h_v(w,u_j^n)dw\Big|$$
$$\le k_{j+\frac{1}{2}}\Big|\int_{u_j^n}^{u_{j+1}^n} |f'(w)|dw\Big|$$

(3.26)

The last inequality follows from 1.9 and 1.11. To estimate this quantity, consider

$$|\Delta_+^u z_j^n| = \frac{k_{j+\frac{1}{2}}}{f^*}|\phi(u_{j+1}^n) - \phi(u_j^n)| = \frac{k_{j+\frac{1}{2}}}{f^*}\Big|\int_{u_j^n}^{u_{j+1}^n} |f'(w)|dw\Big| \tag{3.27}$$

Comparing 3.26 and 3.27 gives

$$k_{j+\frac{1}{2}}|h_{j+\frac{1}{2}} - f(u^\Delta(x_j,t^n))| \le f^*|\Delta_+^u z_j^n| \tag{3.28}$$

Thus the contribution to the left side of 3.24 from the first term in 3.25 is bounded by

$$\Delta x\Delta t\|\psi_x\|_\infty \sum_j \sum_{n=0}^N k_{j+\frac{1}{2}}|h_{j+\frac{1}{2}} - f(u^\Delta(x_j,t^n))| \le \Delta x\Delta t\|\psi_x\|_\infty \sum_j \sum_{n=0}^N f^*|\Delta_+^u z_j^n|$$
$$\le \Delta x\Delta t\|\psi_x\|_\infty f^* \sum_{n=0}^N TV(z^n)$$

(3.29)

This last term tends to zero along with Δx, since $TV(z^n)$ is uniformly bounded, by Theorem 3.4. □

It is now possible to prove Theorem 3.2.

Proof. The CFL condition guarantees that the computed solutions u^Δ remain within $[0,1]$. An application of Proposition 3.1 gives uniform (with respect to both t and Δ) bounds on $\|z^\Delta(\cdot,t)\|_\infty$ and $\|z^\Delta(\cdot,t)\|_{L^1[-B,B]}$, for any compact interval $[-B,B]$. Theorem 3.4 provides a uniform bound on $TV(z^\Delta(\cdot,t))$. Finally, from Lemma 3.5, it follows that

$$\|z^\Delta(\cdot,t+\tau) - z^\Delta(\cdot,t)\|_1 \le C(|\tau| + \Delta) \tag{3.30}$$

where the constant C is independent of Δ and t. By standard compactness arguments [23], [22] applied to the sequence z^Δ, there is a subsequence, z^{Δ_i}, which converges in L^1_{loc} to some function $z \in L^1_{loc}\bigcap L^\infty \bigcap BV$. There is a further subsequence, also denoted z^{Δ_i}, which converges a.e. to z. Let $u(x,t) = \Psi^{-1}(z(x,t),k(x))$. Due to the strict monotonicity of $\Psi(\cdot,k)$, the function u is well-defined a.e. Dropping the subscript on Δ, it remains to show that $u^\Delta \to u$ a.e. and in L^1_{loc}. Using the fact that $u^\Delta = \Psi^{-1}(z^\Delta,k^\Delta)$ gives

$$\int\int |u - u^\Delta|dxdt \le \int\int |\Psi^{-1}(z,k) - \Psi^{-1}(z^\Delta,k)|dxdt$$
$$+ \int\int |\Psi^{-1}(z^\Delta,k) - \Psi^{-1}(z^\Delta,k^\Delta)|dxdt$$

(3.31)

The first integral tends to zero by the bounded convergence theorem, due to the continuity of Ψ^{-1} as a function of its first argument. For the second integral, an estimate of $|\Psi^{-1}(z^\Delta, k) - \Psi^{-1}(z^\Delta, k^\Delta)| = |u(z^\Delta, k) - u(z^\Delta, k^\Delta)|$ is required. Implicit differentiation gives

$$\frac{\partial u}{\partial k} = -\Psi_k/\Psi_u = \frac{-1}{k}\frac{\int_{u^*}^{u} |f'(w)|dw}{|f'(u)|} \tag{3.32}$$

When the numerator and denominator are each expanded in Taylor series about u^*, the result is

$$\Big|\frac{\partial u}{\partial k}\Big| \le \frac{1}{2\underline{k}}\frac{\|f''\|_\infty}{|\mu|}|u - u^*| \tag{3.33}$$

With this estimate, convergence of the second integral follows from 1.13. Having established convergence in L^1_{loc}, there is yet a further subsequence u^Δ, which converges to u, boundedly a.e., and in L^1_{loc}. Theorem 3.6 guarantees that u is a weak solution of the conservation law. ∎

4. Entropy satisfaction. Even for constant $k(x)$, solutions of 1.1 are not necessarily unique. Additional conditions, often derived from physical considerations, and usually referred to as entropy conditions, are required. Uniqueness is guaranteed within the class of weak solutions which additionally satisfy these entropy conditions. When $k(x)$ is smooth the Kruzkov [11] entropy condition applies. More specifically, uniqueness is guaranteed if the following inequality holds for all real numbers c, and for every smooth test function $\psi \ge 0$ with compact support in $t > 0$, $-\infty < x < +\infty$.

$$\int_{t>0}\int_{-\infty}^{+\infty} \big(|u-c|\psi_t + k\sigma(u-c)(f(u)-f(c))\psi_x - \sigma(u-c)k'f(c)\psi\big)dxdt \ge 0 \tag{4.1}$$

In the case where k is constant, the numerical scheme 1.6 is well known to generate approximations which satisfy a discrete version of 4.1 [2], [8]. Via the convergence arguments of the Lax-Wendroff theorem, for constant k, it follows that the limit of any convergent subsequence satisfies 4.1. Unfortunately, the Kruzkov entropy condition does not appear to generalize easily to the nonsmooth k setting, as was observed in [10]. For $k(x)$ discontinuous, the authors of [10] have shown uniqueness, as well as existence and asymptotic behavior, of 1.1 within the class of solutions that satisfy a wave entropy condition. They also showed that, for smooth k, the wave entropy condition gives the same class of weak solutions as 4.1. However, a discrete version of this wave entropy condition is not readily apparent.

The approach in this section is to concentrate on the Godunov version of the scheme, and derive a set of discrete entropy inequalities. The discrete entropy inequalities imply corresponding entropy inequalities for the limit of the computed approximations. The inequalities established for the limit solution u imply, for piecewise smooth u, geometric entropy conditions, involving the characteristics on either side of the discontinuity.

As in [10], assume that $k(x)$ is piecewise smooth, with finitely many jumps at the points $\xi_1 < \xi_2 < \ldots < \xi_M$. Also assume that, for some constant $\beta \ge 0$, $|k'(x)| \le \beta$ for all x. The allowable time step will be reduced so that the CFL condition $2\lambda\|k\|_\infty\|f'\|_\infty \le 1$ holds. For the Godunov version of the scheme, a single time step can be viewed as solving a Riemann problem at each cell boundary (due to jumps in u), and another Riemann problem at each half-cell boundary (due to jumps in

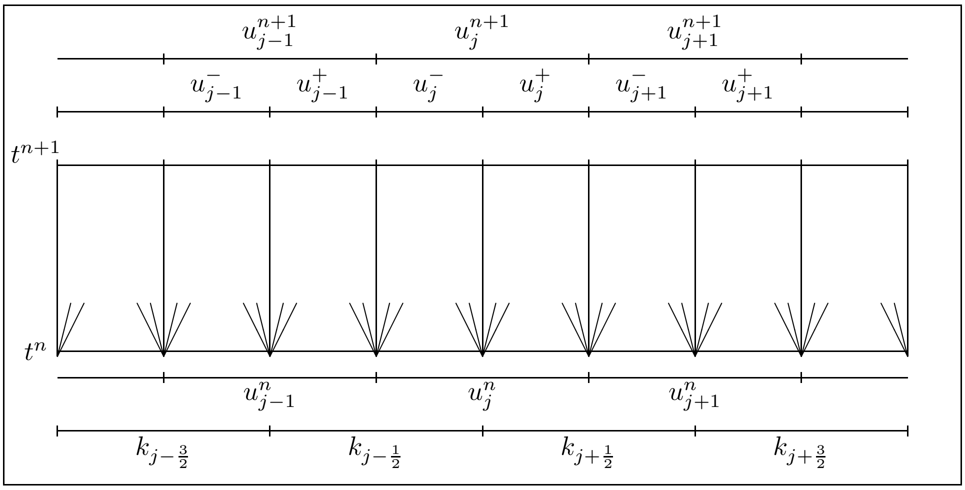

FIG. 4.1. *Riemann problems for half-cells.*

k), and then averaging. The solution operator used at cell boundaries is the usual constant k version, resulting in the usual Godunov flux. The solution operator at half-cell boundaries is the one described in the appendix, which applies to the case when there is a jump in k. In order to derive a discrete entropy inequality for the Godunov version of the scheme, the averaging is performed in two steps, as in [24]. First, the average is taken over half-cells, then the half-cell averages are combined. See Figure 4.1. The reduced time step guarantees that waves arising from the cell and half-cell boundaries do not travel more than a half cell before the end of the time step. Let

$$u_j^{n+1} = \frac{1}{2}u_j^- + \frac{1}{2}u_j^+ \tag{4.2}$$

where the half-cell averages are given by

$$u_j^- = u_j^n - 2\lambda k_{j-\frac{1}{2}}(f(\tilde{u}_L(u_j)) - h_{j-\frac{1}{2}}) \tag{4.3}$$

$$u_j^+ = u_j^n - 2\lambda k_{j+\frac{1}{2}}(h_{j+\frac{1}{2}} - f(\tilde{u}_R(u_j))) \tag{4.4}$$

Here $\tilde{u}_L(u_j)$ ($\tilde{u}_R(u_j)$) is the left (right) intermediate point for the solution of the Riemann problem (see the appendix) with data $(k_{j-\frac{1}{2}}, k_{j+\frac{1}{2}})$ and (u_j^n, u_j^n). That 4.2 holds follows from the Rankine-Hugoniot condition $k_{j-\frac{1}{2}}f(\tilde{u}_L(u_j)) = k_{j+\frac{1}{2}}f(\tilde{u}_R(u_j))$. Let (V, F) be a convex entropy and entropy flux, i.e., V is convex and $F' = V'f'$. Because of the additivity property A.5, $f(\tilde{u}_L(u_j))$ can be viewed as a Godunov flux, $f(\tilde{u}_L(u_j)) = h(\tilde{u}_L(u_j), u_j^n)$, and so the following discrete entropy inequality holds.

$$V(u_j^-) \le V(u_j^n) - 2\lambda k_{j-\frac{1}{2}}(F(\tilde{u}_L(u_j)) - H_{j-\frac{1}{2}}) \tag{4.5}$$

Similarly, $f(\tilde{u}_R(u_j))$ can be viewed as a Godunov flux, $f(\tilde{u}_R(u_j)) = h(u_j^n, \tilde{u}_R(u_j))$ which yields

$$V(u_j^+) \le V(u_j^n) - 2\lambda k_{j+\frac{1}{2}}(H_{j+\frac{1}{2}} - F(\tilde{u}_R(u_j))) \tag{4.6}$$

Here, the numerical entropy flux $H_{j+\frac{1}{2}}$ is defined by $H_{j+\frac{1}{2}} = F(u^G_{j+\frac{1}{2}})$, i.e, the numerical entropy flux normally associated with the scalar Godunov scheme for constant k [15]. Multiplying both 4.5 and 4.6 by 1/2, adding, and applying Jensen's inequality, gives

$$V(u_j^{n+1}) \le V(u_j^n) - \lambda(k_{j+\frac{1}{2}}H_{j+\frac{1}{2}} - k_{j-\frac{1}{2}}H_{j-\frac{1}{2}}) + \lambda(k_{j+\frac{1}{2}}F(\tilde{u}_R(u_j)) - k_{j-\frac{1}{2}}F(\tilde{u}_L(u_j))) \tag{4.7}$$

Now, let $V(u) = |u-c|$, and $F(u) = \sigma(u-c)(f(u)-f(c))$, where c is any real number. Proposition A.4 gives

$$k_{j+\frac{1}{2}}F(\tilde{u}_R(u_j)) - k_{j-\frac{1}{2}}F(\tilde{u}_L(u_j)) \le |k_{j+\frac{1}{2}} - k_{j-\frac{1}{2}}|f(c) \tag{4.8}$$

The discrete entropy inequality 4.7, 4.8 results in an entropy inequality for the limit u of the computed approximations.

Proposition 4.1. *In addition to the conditions stated in Theorem 3.2, let k be piecewise smooth, with a bounded derivative, $|k'(x)| \le \beta$ for all x, and with finitely many jumps (in k) , located at $\xi_1 < \xi_2 < \ldots < \xi_M$. Assume that the allowable time step is reduced so that the CFL condition $2\lambda\|k\|_\infty\|f'\|_\infty \le 1$ holds. Let u^Δ be a convergent subsequence generated by the scheme 1.6 using the Godunov flux, converging to u, with $TV(z^\Delta)$ uniformly bounded, as in Theorem 3.2. For every smooth test function $\psi \ge 0$ with compact support in $t > 0$, $-\infty < x < +\infty$, and every $c \in \mathbf{R}$, the following inequality holds*

$$\begin{aligned}
&\int_{t>0}\int_{-\infty}^{+\infty} \big(|u-c|\psi_t + k\sigma(u-c)(f(u)-f(c))\psi_x)dxdt \\
&+ \int_{t>0}\int_{-\infty}^{+\infty} \sum_{i=1}^{M} |k(\xi_i^+) - k(\xi_i^-)|\delta(x-\xi_i)f(c)\psi dxdt \\
&+ \int_{t>0}\int_{-\infty}^{+\infty} |k'(x)|f(c)\psi dxdt \ge 0
\end{aligned} \tag{4.9}$$

where $\delta(x-\xi_i)$ is the Dirac measure with support located at ξ_i.

Proof. Multiplying 4.7 by a smooth, nonnegative test function ψ with compact support in $t > 0$, using 4.8, and proceeding as in the proof of Theorem 3.6 gives

$$\begin{aligned}
&-\Delta x\Delta t\sum_{j,n} V_j^n(\Delta^t\psi_j^n/\Delta t) - \Delta x\Delta t\sum_{j,n} k_{j+\frac{1}{2}}H_{j+\frac{1}{2}}(\Delta^x\psi_j^n/\Delta x) \\
&-\Delta x\Delta t\sum_{j,n}\psi_j^n f(c)(\lambda/\Delta t)|\Delta_+ k_{j-\frac{1}{2}}| \le 0
\end{aligned} \tag{4.10}$$

As Δ approaches zero, the first sum in the top line of 4.10 converges to $-\int\int V(u)\psi_t$, as in the proof of the Lax-Wendroff theorem. Convergence of the second sum is proved in a manner similar to the proof of 3.23 in the proof of Theorem 3.6. In fact, by a similar arguments, the proof here boils down to estimating the quantity $|H_{j+\frac{1}{2}} - F(u^\Delta(x_j,t^n)|$. From the definitions, and using the fact that $u^G_{j+\frac{1}{2}}$ lies between u_j^n and u_{j+1}^n,

$$|H_{j+\frac{1}{2}} - F(u^\Delta(x_j,t^n))| = \Big|\int_{u_j^n}^{u^G_{j+\frac{1}{2}}} V'(w)f'(w)dw\Big|$$

$$\leq \|V'\|_\infty \Big| \int_{u_j^n}^{u^G_{j+\frac{1}{2}}} |f'(w)|dw \Big|$$
$$\leq \Big| \int_{u_j^n}^{u_{j+1}^n} |f'(w)|dw \Big| \tag{4.11}$$

Convergence of the second sum in the top line of 4.10 then follows as in the proof of Theorem 3.23. The term in the bottom line of 4.10 converges to the sum of the integrals in the bottom two lines of 4.9. This can be verified by breaking the spatial portion of the sum in the bottom line of 4.10 into sums over intervals where k is smooth, and where k has its jumps. □

The entropy inequalities 4.9 yield the following proposition concerning the satisfaction of geometric entropy conditions.

PROPOSITION 4.2. *Assume, in addition to the assumptions of 4.1 that the limit solution u is piecewise smooth. Then any discontinuity in u at a jump in k satisfies the following geometric entropy condition:*

$$\textit{Either } f'(u_L) \geq 0 \textit{ or } f'(u_R) \leq 0 \textit{ or both.} \tag{4.12}$$

Any discontinuity located away from a jump in k satisfies

$$kf'(u_L) > s > kf'(u_R) \tag{4.13}$$

where s is the shock speed, defined by $s = k(f(u_R) - f(u_L))/(u_R - u_L)$.

Proof. For 4.12, assume, by way of contradiction, that there is a neighborhood N_i, which intersects the line $x = \xi_i$, and that, within N_i, $f'(u(x,t)) > 0$ for $x > \xi_i$, $f'(u(x,t)) < 0$ for $x < \xi_i$. Assume that k jumps from $k_L = k(\xi_i^-)$ to $k_R = k(\xi_i^+)$ across the line $x = \xi_i$, and that u is smooth in N_i, except for the jump at $x = \xi_i$. It is shown in Proposition A.3 that if 4.12 fails, then, with $F(u) = \sigma(u - u^*)(f(u) - f^*)$, $k_R F(u_R) - k_L F(u_L) > |k_R - k_L| f^*$. Let $\psi^\epsilon \geq 0$ be a smooth test function with support in the rectangle Ω^ϵ centered at (x_0, t_0), with width ϵ, and extending backward and forward in time from t_0 by an amount τ. Assume that ϵ and τ are small enough that $\Omega^\epsilon \subset N_i$. A standard test function calculation, applied to 4.9, gives

$$\int_{t_0-\tau}^{t_0+\tau} \psi^\epsilon(\xi_i, t)[kF]dt - f^* \int_{t_0-\tau}^{t_0+\tau} \psi^\epsilon(\xi_i, t)|k_R - k_L|dt$$
$$\leq f(c) \int_{\Omega^\epsilon} \{|k'(x)| - \sigma(u - c)k'(x)\} \psi^\epsilon(x,t)dxdt \tag{4.14}$$

where the notation $[A]$ indicates the jump in the quantity A, $[A] = A(x^+, t) - A(x^-, t) = A_R - A_L$. The integral over Ω^ϵ can be made arbitrarily small by shrinking the width ϵ of the rectangle. It follows that $k_R F(u_R) - k_L F(u_L) \leq |k_R - k_L| f(u^*)$, which is the desired contradiction.

For 4.13, suppose that a smooth shock curve $x(t)$ runs through a point (x_0, t_0) located away from any jumps in k. That $\dot{x} = s$ is a well-known consequence of the fact that u satisfies 1.3. Let $\psi^\epsilon \geq 0$ be a smooth test function with support in the tube Ω^ϵ centered at the shock curve, with width ϵ, and extending backward and forward

in time from t_0 by an amount τ. With $V(u) = |u - c|$, $F(u) = \sigma(u - c)(f(u) - f(c))$, a standard test function calculation applied to 4.9 gives

$$\int_{t_0-\tau}^{t_0+\tau} \psi^\epsilon(x(t), t)([kF] - s[V])dt \\ \le f(c) \int_{\Omega^\epsilon} \{|k'(x)| - \sigma(u - c)k'(x)\}\psi^\epsilon(x, t)dxdt \tag{4.15}$$

The integral on the right of 4.15 can be made arbitrarily small by shrinking the width ϵ of Ω^ϵ. It follows that

$$k(x_0)(F(x_0^+, t_0) - F(x_0^-, t_0)) - s(V(x_0^+, t_0) - V(x_0^-, t_0)) \le 0 \tag{4.16}$$

It is well known [11] that if 4.16 holds for all $c \in \mathbf{R}$, then the geometric entropy condition 4.13 is satisfied. ∎

The geometric entropy condition 4.13 is the usual geometric entropy condition satisfied by shocks in the smooth k case. For smooth k, the entropy inequality 4.1 implies 4.13 for piecewise smooth solutions [11]. It requires that characteristics on both sides of the shock extend toward the x-axis when followed backward in time from the shock. The geometric entropy condition 4.12 requires that the characteristics on at least one side extend toward the x-axis. (See Figure A.1.) The case where both characteristics extend toward the x-axis corresponds to a shock located at the jump in k. The other cases are simply jumps in the solution, due to the jump in k, similar to the situation that results from a linear PDE with a jump in the coefficient. In the appendix, it is shown that the two entropy conditions 4.12 and 4.13 are sufficient to guarantee uniqueness for the Riemann problem, where both u_0 and k are constant, except for a single jump, in both u_0 and k, located at $x = 0$.

Theorem 4.4 below ensures that when k is smooth, the Kruzkov entropy inequalities 4.1 are satisfied by the limit solution u, and it is not necessary to halve the allowable time step. In order to avoid technical difficulties arising from the discontinuity in $V'(u) = \sigma(u - c)$, the following entropy inequality is first established for smooth V.

Lemma 4.3. *In addition to the conditions stated in Theorem 3.2, let k be smooth, with a bounded derivative, $|k'| \le \beta$. Let (V, F) be a convex entropy pair for 1.1, i.e., V is convex and $F' = V'f'$, and assume that $V \in C^2[0, 1]$. Let u^Δ be a convergent subsequence generated by the scheme 1.6 using the Godunov flux, converging to u, with $TV(z^\Delta)$ uniformly bounded, as in Theorem 3.2. The following entropy condition holds*

$$\int_{t>0} \int_{-\infty}^{+\infty} (V(u)\psi_t + kF(u))\psi_x)dxdt - \int_{t>0} \int_{-\infty}^{+\infty} k'(V'(u)f(u) - F(u))\psi dxdt \ge 0 \tag{4.17}$$

for every smooth test function $\psi \ge 0$ with compact support in $t > 0$, $-\infty < x < +\infty$.

Proof. Write the scheme 1.6 as

$$u_j^{n+1} = w_j^n - \lambda h_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}} \tag{4.18}$$

where $w_j^n = u_j^n - \lambda(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k_{j+\frac{1}{2}} h_{j-\frac{1}{2}})$. The following discrete entropy inequality holds for w_j^n:

$$V(w_j^n) \le V(u_j^n) - \lambda(k_{j+\frac{1}{2}} H_{j+\frac{1}{2}} - k_{j+\frac{1}{2}} H_{j-\frac{1}{2}}) \tag{4.19}$$

where, as before, the Godunov discrete entropy flux is defined by $H_{j+\frac{1}{2}} = F(u^G_{j+\frac{1}{2}})$. After rearranging terms, the following discrete entropy inequality results.

$$\begin{aligned} V_j^{n+1} \le\ & V_j^n - \lambda(k_{j+\frac{1}{2}} H_{j+\frac{1}{2}} - k_{j-\frac{1}{2}} H_{j-\frac{1}{2}}) \\ & + (V_j^{n+1} - V(w_j^n)) + \lambda H_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}} \end{aligned} \tag{4.20}$$

Multiplying by a smooth, nonnegative test function ψ, and proceeding as in the proof of Theorem 4.1 gives

$$\begin{aligned} & - \Delta x \Delta t \sum_{j,n} V_j^n (\Delta^t \psi_j^n / \Delta t) - \Delta x \Delta t \sum_{j,n} k_{j+\frac{1}{2}} H_{j+\frac{1}{2}} (\Delta^x \psi_j^n / \Delta x) \\ & - \Delta x \Delta t \sum_{j,n} \psi_j^n \Big(H_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}} / \Delta x) + (V_j^{n+1} - V(w_j^n))/\Delta t \Big) \le 0 \end{aligned} \tag{4.21}$$

The arguments used in the proof of Theorem 4.1 apply to the sums in the top line of 4.21. Consider the bottom line of 4.21. Using the fact that k is smooth, and the bound on the variation of z^Δ again,

$$\Delta x \Delta t \sum_{j,n} \psi_j^n H_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}} / \Delta x) \to \int_{t>0} \int_{-\infty}^{+\infty} \psi F(u) k' \, dx dt \tag{4.22}$$

The last term to be dealt with is $\Delta x \Delta t \sum_{j,n} \psi_j^n (V_j^{n+1} - V(w_j^n))/\Delta t$. Expanding the divided difference in a Taylor series gives

$$\begin{aligned} (V_j^{n+1} - V(w_j^n))/\Delta t = & - V'(u_j^n) h_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}} / \Delta x) \\ & - \frac{1}{2} V''(\theta_1) (\lambda h_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}})^2 / \Delta t \\ & - V''(\theta_2) h_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}} / \Delta x)(u_j^{n+1} - u_j^n) \end{aligned} \tag{4.23}$$

Due to the smoothness of k, L^1-contractiveness, and the time continuity estimate 3.18, the sums involving the last two terms on the right side of 4.23 approach zero as $\Delta \to 0$. The first term on the right side of 4.23 approaches $-V'(u) f(u) k'$, which gives

$$\Delta x \Delta t \sum_{j,n} \psi_j^n (V_j^{n+1} - V(w_j^n))/\Delta t \to - \int_{t>0} \int_{-\infty}^{+\infty} \psi V'(u) f(u) k' \, dx dt \tag{4.24}$$

□

Theorem 4.4. *In addition to the conditions of Theorem 3.2, let k be continuously differentiable, with a bounded derivative, $|k'| \le \beta$. Let u^Δ be a convergent subsequence generated by the scheme 1.6 using the Godunov flux, converging to u, with $TV(z^\Delta)$ uniformly bounded, as in Theorem 3.2. Then the Kruzkov entropy inequalities 4.1 hold for every real number c, and so the whole sequence u^Δ converges to the unique entropy solution of 1.3.*

Proof. As in [11], approximate the convex function $|u-c|$ by a sequence of twice continuously differentiable convex functions V_i. Apply Lemma 4.3 for each V_i, and let $V_i \to |u-c|$. □

5. The Lax-Friedrichs version of the scheme. The Lax-Friedrichs (LF) numerical flux is given by

$$h(v,u) = \frac{1}{2}(f(u)+f(v)) - \frac{q}{2\lambda}(v-u) \tag{5.1}$$

where the parameter $q \in (0,1]$ is the numerical viscosity. It is monotone and as smooth as the flux $f(u)$. The LF flux is centered, as opposed to the Godunov and EO fluxes, which are one-sided, or upwind fluxes. An important advantage of the LF flux is the simplicity that results from being centered. The main disadvantage is that the resulting scheme is very diffusive, which gives shocks that are smeared over many mesh points. The analog of 1.6 for the LF flux is

$$u_j^{n+1} = u_j^n - \lambda(h^{LF}_{j+\frac{1}{2}} - h^{LF}_{j-\frac{1}{2}}) \tag{5.2}$$

where

$$h^{LF}_{j+\frac{1}{2}} = \frac{1}{2}k_{j+\frac{1}{2}}(f(u_j)+f(u_{j+1})) - \frac{q}{2\lambda}(u_{j+1}-u_j) \tag{5.3}$$

It is easily verified that the results of Propositions 2.1 and 2.2 hold for the LF version of the scheme 5.2 with the CFL condition $\lambda\|k\|_\infty\|f'\|_\infty \le \min(q, 1-q)$. Unlike the upwind versions of the scheme, the CFL condition does not have to be modified when zero crossings of $k(x)$ are present. However, $\min(q,1-q) \le 1/2$, and the size of the allowable time step approaches zero as q approaches 1. Another way to define the LF flux in the variable k situation is

$$g^{LF}_{j+\frac{1}{2}} = \frac{1}{2}(k_j f(u_j) + k_{j+1}f(u_{j+1})) - \frac{q}{2\lambda}(u_{j+1}-u_j) \tag{5.4}$$

where k_j is consistent with a discretization of $k(x)$ that is aligned with that of u_0 :

$$k^\Delta(x) = \sum_j \chi_j(x)k_j \tag{5.5}$$

With this approach, the results of Propositions 2.1 and 2.2 hold with the CFL condition $\lambda\|k\|_\infty\|f'\|_\infty \le 1$, and no reduction for zero crossings of k is required. The scheme that results with 5.5 is closely related to the version of the Lax-Friedrichs scheme discussed in [21], with the exception that the scheme discussed there is the 2x2 system version, so that k^Δ is not constant in time.

6. More general spatial dependence. This paper has focused on equations of the form 1.1, where the spatial dependence appears as a coefficient of the flux f in the form $k(x)f(u)$. A more general Cauchy problem is

$$u_t + f(u,k)_x = 0, \quad u(x,0) = u_0(x) \tag{6.1}$$

Equations of this type arise in models of two-phase flow in porous media [9]. The more general form of the difference scheme for 6.1 is

$$u_j^{n+1} = u_j^n - \lambda(h^k_{j+\frac{1}{2}} - h^k_{j-\frac{1}{2}}) \tag{6.2}$$

Here, the superscript k denotes the dependence of the flux on $k^\Delta(x)$: $h^k_{j+\frac{1}{2}} = h(u_{j+1}, u_j; k_{j+\frac{1}{2}})$. The Godunov version of the numerical flux for this scheme is

$$h(v,u;k) = \begin{cases} \min_{[u,v]} f(w,k) & \text{if } u \le v \\ \max_{[v,u]} f(w,k) & \text{if } u \ge v \end{cases} \tag{6.3}$$

and the EO version of the numerical flux is

$$h(v,u;k) = \frac{1}{2}(f(u,k)+f(v,k)) - \frac{1}{2}\int_u^v |f_w(w,k)|dw \tag{6.4}$$

The following fact results from a slight generalization of the proof of Proposition 2.1.

PROPOSITION 6.1. *With the CFL condition $2\lambda\|f_u\|_\infty \leq 1$ both the Godunov and EO versions of the scheme 6.1 are monotone. If the initial data $u_0(x)$ lies within the interval $\Omega_0 = [u_{min}, u_{max}]$, and $f(u_{min}, k) = a$ for all k, $f(u_{max}, k) = b$ for all k, then the computed solution will remain within Ω_0. If, for each $u \in [u_{min}, u_{max}]$, the sign of $f_u(u,k)$ is constant over the range of $k(x)$, the CFL condition can be relaxed to $\lambda\|f_u\|_\infty \leq 1$.*

Proposition 2.2, assuring L^1-contractiveness, also carries over to this situation, assuming that $u_0,\ v_0 \in L^1 \bigcap L^\infty$, and $k \in L^\infty$ and is constant for large x.

7. Examples. Several examples are presented in this section.

7.1. Avoidance of spurious total variation blowup. The following example is given in [21]: $f(u) = u(1-u)$, $k(x) = x$, and $u_0(x) = \frac{1}{2}$. It provides an application of the scheme 1.6 when the coefficient $k(x)$ is allowed to cross zero. The unique solution of this problem is $u(x,t) = (1+e^t)^{-1}$, which is independent of the spatial variable x, and thus has zero total variation. When the coefficient $k(x)$ is discretized on the same mesh as the conserved quantity u, i.e., not staggered, the resulting Riemann problems that must be solved at each cell interface are more complex. In addition, as shown in [21] , as the mesh shrinks, the variation of the first step of the solution computed in this way blows up. For the 2x2 Godunov scheme, this is not a problem, since averaging suppresses the oscillations, as shown in [17]. However, these oscillations make it impossible to continue the random choice method or front tracking scheme. It is not hard to check that the scalar schemes presented in this paper result in a computed solution to this problem which is independent of the spatial variable, like the true solution, and so the variation is zero. In fact, all three (Godunov, EO, and LF) versions of the scheme give $u^{n+1} = u^n - \Delta t f(u^n)$. Letting $\Delta t \to 0$, the computed solution tends to the solution of the O.D.E. $\dot{u} = -u(1-u)$, whose solution, with the given initial condition, is $(1+e^t)^{-1}$. In fact, since the initial data U^0 is constant (and by induction so is the computed solution at each time step u^n), $h_{j+\frac{1}{2}} = h(u^n, u^n) = f(u^n)$. As pointed out in [21], the 2x2 system version of the Lax-Friedrichs scheme, with an aligned discretization of k, also gives the correct solution for this example.

7.2. Avoidance of Riemann problems with no solution. The next example is also taken from [21]. Here, $f(u) = 1 + u^2/(1+u^2)$, $k(x) = x$, and $u_0(x) = 0$. As before this problem has a unique solution, which is independent of the spatial variable x. Specifically it is the solution of the O.D.E. $\dot{u} = -f(u)$. As in the previous example, problems arise when the coefficient $k(x)$ is discretized on the same mesh as the conserved quantity u. In particular, if a cell boundary coincides with the point $x = 0$, the Riemann problem that results has no solution. With the staggered mesh approach, and any two-point consistent numerical flux, the resulting computed solution is independent of x, and satisfies $u^{n+1} = u^n - \Delta t f(u^n)$, which is Euler's method for the O.D.E. $\dot{u} = -f(u)$. Again, the 2x2 version of the Lax-Friedrichs scheme, with an aligned discretization of k, also gives the correct solution for this example [21].

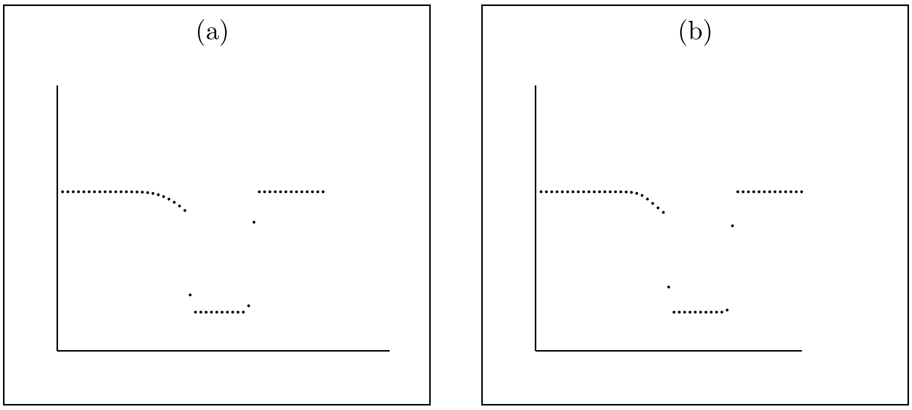

FIG. 7.1. *Numerical example 1. (a) Godunov version. (b) Godunov version with flux-limited correction terms.*

7.3. Numerical example 1. Figure 7.1 shows the result of two runs of the scheme with the Riemann problem having constant initial data $(u_L, u_R) = (0.6, 0.6)$, and the coefficient given by $(k_L, k_R) = (0.05, 0.1)$. The Godunov version of the numerical flux was used, with (b) and without (a) flux-limited second order correction terms, and with the flux $f(u) = u(1-u)$. Although this paper has not emphasized schemes with higher order accuracy, this example provides some evidence that the approach presented here can be used as a starting point for constructing more accurate algorithms. Each scheme was run for 50 time steps, with spatial and time spacing of $\Delta x = .02$ and $\Delta t = .19$. This allowed for a maximum Courant number of 0.95 in case the solution approached $u = 0$ or $u = 1$, where f' assumes its maximum. The maximum Courant number actually achieved during the run was approximately .57. With respect to the enumeration of Riemann problem solutions in the appendix, the solution corresponds to Case 1, $r_L d_0 s_R$. In this example, the discretizations of the data were such that k^Δ had an intermediate state between k_L and k_R, while u_0^Δ had no intermediate state.

7.4. Numerical example 2. Figure 7.2 shows the result of two runs of the scheme 1.6 with the Riemann problem having initial data $(u_L, u_R) = (0.2, 0.6)$, and the coefficient given by $(k_L, k_R) = (0.1, -0.05)$. This provides an example where k is allowed to take both signs. The Godunov (a) and LF (b) (version 5.3, with $q = 1/2$) versions of the numerical flux were used, with the flux $f(u) = u(1-u)$. The scheme was run for 50 time steps, with spatial and time spacing of $\Delta x = .02$ and $\Delta t = .1$. This allowed for a maximum Courant number of 0.5 in case the solution approached $u = 0$ or $u = 1$, where f' assumes its max. In this case, the CFL limit was achieved. From left to right, the solution consists of the constant state u_L, connected to the intermediate state $u_M = 1$ by a shock moving to the left, which is connected to the constant state u_R by a rarefaction moving to the right. At the location of the jump in k, the solution is continuous, but the characteristics are not. As in the previous example, the discretizations were such that k^Δ had an intermediate state between k_L and k_R, while u_0^Δ had no intermediate state.

7.5. Numerical example 3. Figure 7.3 shows the result of two runs of the scheme 1.6 with the Riemann problem having initial data $(u_L, u_R) = (0.9, 0.0472307)$, and the coefficient given by $(k_L, k_R) = (0.05, 0.1)$. For this example, the initial data

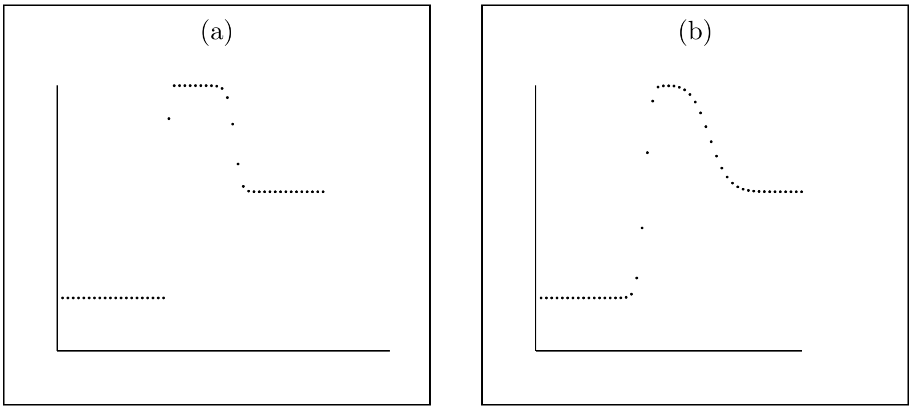

FIG. 7.2. *Numerical example 2. (a) Godunov version. (b) LF version.*

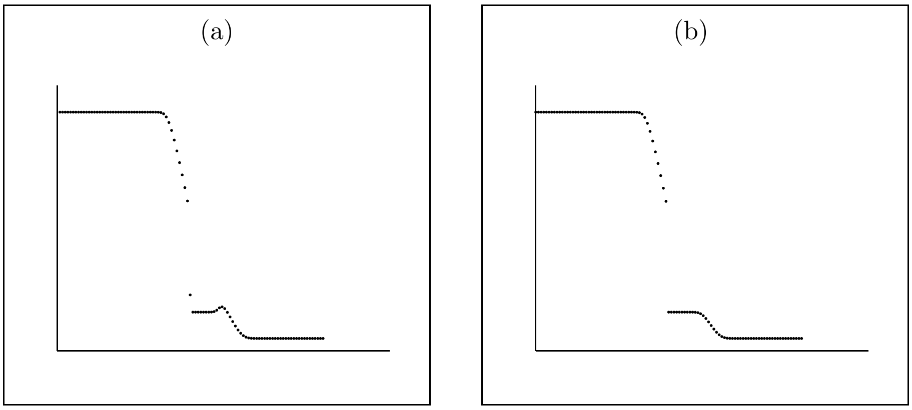

FIG. 7.3. *Numerical example 3. (a) $k(x)$ discretized with an intermediate state between k_L and k_R. (b) $k(x)$ discretized with no intermediate state.*

u_0 is a steady entropy-violating weak solution of the conservation law. It satisfies the Rankine-Hugoniot condition $k_L f(u_L) = k_R f(u_R)$, but violates any reasonable entropy condition, since $f'(u_L) < 0 < f'(u_R)$. The Godunov version of the flux was used, with the flux $f(u) = u(1-u)$. The scheme was run for 400 time steps, with spatial and time spacing of $\Delta x = .0025$ and $\Delta t = .005$. The scheme was run with two different discretizations of k. In (a), the discretization of k had an intermediate state, and the discretization of u_0 did not, as in the previous two numerical examples. In (b), a slight change was made to the discretization of k. This time there was no intermediate state. The jump in k was located a half cell to the right of the jump in u_0. With respect to the various Riemann problem solutions discussed in appendix, the solution corresponds to Case 4, $r_L d_0 r_R$. The initial data lies on the curve segment labeled D in Figure A.2. In (a), the small bump at the edge of the rarefaction on the right is a numerical artifact. As expected, it diminishes in size as the spatial mesh shrinks. The bump forms shortly after $t = 0$, and then moves to the right at the edge of the rarefaction. It is clearly related to the particular discretization of k, since it is not present in (b).

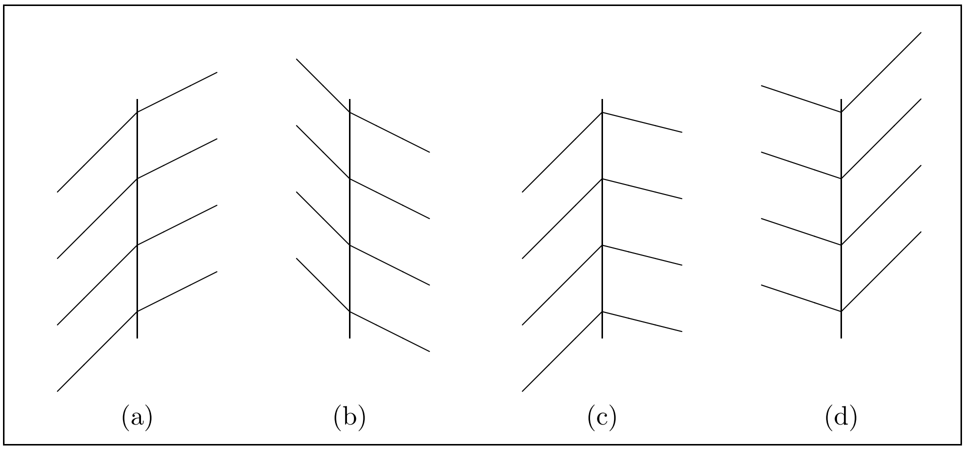

Fig. A.1. *Geometric entropy condition for the characteristics ($\dot{x} = kf'$) at a jump in k. (a), (b) Admissible discontinuities. (c) Admissible shock. (d) Entropy violation.*

Appendix. Entropy satisfaction for the Riemann problem. This section establishes entropy related properties of the solution of the Riemann problem which are required in Section 4. In the process, a construction of the solution is presented. Reference [9] provides a construction of the solution to the Riemann problem within the 2x2 system framework, by connecting right and left states in (u, k) space, and by using a minimization principle to single out a unique solution. The approach here is more within the scalar framework of this paper, making it easier to derive those properties of the solution which are of interest in the scalar context.

The Riemann problem is a special case of the Cauchy problem 1.1, in which the initial data $u_0(x)$ and the coefficient $k(x)$ are constants, except for jumps at $x = 0$. The left and right values of $u_0(x)$ and $k(x)$ are (u_L, u_R) and (k_L, k_R) respectively. First, assume $0 < k_L \leq k_R$; the case where $k_L \geq k_R > 0$ is similar, so the detailed construction is only included for $k_L \leq k_R$. The notation $\tilde{u}_L = u(0^-, t) = u(0^-)$, $\tilde{u}_R = u(0^+, t) = u(0^+)$, for the right and left intermediate states, will be used. With this notation the Rankine-Hugoniot condition across the jump at $x = 0$ is

$$k_L f(\widetilde{u}_L) = k_R f(\widetilde{u}_R) \tag{A.1}$$

The following notation will be useful in describing the solution to the Riemann problem. A shock moving to the right or left is denoted s_R or s_L, a rarefaction is r_R or r_L, a discontinuity at $x = 0$ is d_0, and a shock at $x = 0$ is s_0. The difference between a shock and discontinuity at $x = 0$ is that for a shock $f'(\widetilde{u}_L) > 0 > f'(\widetilde{u}_R)$, but not for a discontinuity. The situation here is slightly different from the constant $k(x)$ setting, where the only allowable discontinuities are shocks. At $x = 0$, there are admissible discontinuities other than shocks. The only ones that are ruled out are ones satisfying $f'(\widetilde{u}_L) < 0 < f'(\widetilde{u}_R)$, where characteristics on both sides of the discontinuity emanate away from the x-axis with increasing time. See Figure A.1. In the process of enumerating the solutions to the Riemann problem, the following fact will be verified.

Proposition A.1. *Let $\tilde{u}_L$ and $\tilde{u}_R$ denote the right and left intermediate states of the solution of the Riemann problem constructed below. The following geometric*

entropy condition holds:

$$\text{(A.2)} \qquad \textit{Either } f'(\tilde{u}_L) \geq 0 \textit{ or } f'(\tilde{u}_R) \leq 0 \textit{ or both.}$$

For a discontinuity located away from the jump in k, with left and right states u^- and u^+, respectively, the usual geometric entropy condition (condition E) holds

$$\text{(A.3)} \qquad kf'(u^-) > s > kf'(u^+), \quad s = k\frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

The construction of the Riemann problem for $0 < k_L \leq k_R$ follows. There are four cases, determined by the location of u_L and u_R with respect to u^*. The two trivial cases where $(u_L, u_R) = (0, 0)$ or $(u_L, u_R) = (1, 1)$ have constant solutions, and will not be considered further.

Case 1: $u_L \geq u^*$, $u_R \geq u^*$. If $k_R f(u_R) \leq k_L f^*$, $\widetilde{u}_R = u_R$, and u_L is connected to an intermediate state $\widetilde{u}_L$ via a wave progressing to the left. The intermediate state $\widetilde{u}_L$ is defined by A.1, along with the condition $\widetilde{u}_L \geq u^*$. The fact that the wave progresses to the left is due to the fact that $\tilde{u}_L$ and u_L lie to the right of u^*. The wave will be a shock if $\widetilde{u}_L > u_L$, and a rarefaction otherwise. With the notation described above, this solution is $s_L d_0$, d_0, or $r_L d_0$, and A.2 holds, since $f'(\tilde{u}_R) \leq 0$, $f'(\tilde{u}_L) \leq 0$. When $k_R f(u_R) > k_L f^*$, a more complicated solution is required. In that case, $\widetilde{u}_L = u^*$, and $\widetilde{u}_R$ is defined via A.1, and $\widetilde{u}_R \leq u^*$. Then u_L is connected to $\widetilde{u}_L$ via a rarefaction progressing to the left, u_R is connected to $\widetilde{u}_R$ via a shock moving to the right. This solution is $r_L d_0 s_R$. Again A.2 holds, since $f'(\tilde{u}_R) \geq 0$, $f'(\tilde{u}_L) = 0$.

Case 2: $u_L \leq u^*$, $u_R \geq u^*$. If $k_L f(u_L) = k_R f(u_R)$, then A.1 is already satisfied with $\tilde{u}_L = u_L$, $\tilde{u}_R = u_R$. This corresponds to a steady shock located at $x = 0$, denoted by s_0, i.e. the initial data is the solution to the Riemann problem, and A.2 holds with $f'(\tilde{u}_R) < 0 < f'(\tilde{u}_L)$. If $k_L f(u_L) > k_R f(u_R)$, then $\tilde{u}_R = u_R$ and $\tilde{u}_L$ is defined via A.1, along with $\widetilde{u}_L \geq u^*$. Then u_L is connected to $\tilde{u}_L$ via a shock progressing to the left. The solution is denoted $s_L d_0$, and A.2 holds, since $f'(\tilde{u}_R) \leq 0$, $f'(\tilde{u}_L) \leq 0$. If $k_L f(u_L) < k_R f(u_R)$, then $\tilde{u}_L = u_L$ and $\tilde{u}_R$ is defined via A.1, along with $\tilde{u}_R \leq u^*$. Then u_R is connected to $\tilde{u}_R$ via a shock progressing to the right. The solution is denoted $d_0 s_R$, and A.2 holds, since $f'(\tilde{u}_R) \geq 0$, $f'(\tilde{u}_L) \geq 0$.

Case 3: $u_L \leq u^*$, $u_R \leq u^*$. In this case, $\widetilde{u}_L = u_L$, and u_R is connected to an intermediate state $\widetilde{u}_R$ via a wave progressing to the right. If $\tilde{u}_R < u_R$, the wave is a shock; if the inequality is reversed, the wave is a rarefaction. The intermediate state $\widetilde{u}_R$ is defined by A.1, and $\widetilde{u}_R \leq u^*$. The fact that the wave progresses to the right is due to the fact that both $\widetilde{u}_R$ and u_R lie to the left of u^*. The solution is denoted $d_0 s_R$, d_0, or $d_0 r_R$. Again, A.2 holds, since $f'(\tilde{u}_R) \geq 0$, $f'(\tilde{u}_L) \geq 0$

Case 4: $u_L \geq u^*$, $u_R \leq u^*$ If $k_R f(u_R) \geq k_L f^*$, two waves are required, one moving in each direction. Let $\tilde{u}_L = u^*$. Define $\tilde{u}_R$ via A.1, along with $\tilde{u}_R \leq u^*$. Then u_L is connected to $\widetilde{u}_L$ via a rarefaction progressing to the left, u_R is connected to $\widetilde{u}_R$ via a shock moving to the right. This solution is denoted $r_L d_0 s_R$. The condition A.2 holds in this case, since $f'(\tilde{u}_R) \geq 0$, $f'(\tilde{u}_L) = 0$. If $k_R f(u_R) < k_L f^*$, the solution is slightly different. Let $\tilde{u}_L = u^*$. Again define $\tilde{u}_R$ via A.1, along with $\tilde{u}_R \leq u^*$. Then u_L is connected to $\widetilde{u}_L$ via a rarefaction progressing to the left, u_R is connected to $\widetilde{u}_R$ via a rarefaction moving to the right. This solution is denoted $r_L d_0 r_R$. The condition A.2 holds in this case also, since $f'(\tilde{u}_R) \geq 0$, $f'(\tilde{u}_L) = 0$.

In all cases where two states are connected by a shock, either u_L is connected to $\tilde{u}_L$ by a shock, or u_R is connected to $\tilde{u}_R$ by a shock. For shocks on the left, the

following inequalities are valid (see Figure A.2)

$$\frac{f(\tilde{u}_L) - f(u_L)}{\tilde{u}_L - u_L} < 0, \quad u_L < \tilde{u}_L \tag{A.4}$$

Similarly, for a shock on the right

$$\frac{f(\tilde{u}_R) - f(u_R)}{\tilde{u}_R - u_R} < 0, \quad u_R > \tilde{u}_R \tag{A.5}$$

For a concave function, the inequalities $u_L < \tilde{u}_L$, $u_R > \tilde{u}_R$ imply A.3.

For $k_L > k_R$, the construction is similar. Case 1 of $k_L > k_R$ is similar to case 3 of $k_L < k_R$, except that the waves move to the left, the solution being $s_L d_0$, d_0, or $r_L d_0$. Case 2 is unchanged. Case 3 of $k_L > k_R$ is similar to case 1 of $k_L < k_R$, with the solution being one of $d_0 s_R$, d_0, $d_0 r_R$, or $s_L d_0 r_R$. Case 4 of $k_L > k_R$ is similar to case 4 of $k_L < k_R$, except that the waves switch sides. The solution is $s_L d_0 r_R$ or $r_L d_0 r_R$. Figure A.2 shows the mapping $(u_L, u_R) \mapsto (\tilde{u}_L, \tilde{u}_R)$ for the case $k_L < k_R$.

The following proposition shows that there is essentially only one way to construct a solution to the Riemann problem such that the geometric entropy conditions stated in Proposition A.1 are satisfied. The proof is an adaptation to the present situation of the proof in [20], where L^1-contractiveness is established under the assumptions that k is constant, the initial data is piecewise smooth, and A.3 is satisfied.

Proposition A.2. *Let u and v be piecewise smooth solutions of the Riemann problem which also satisfy the geometric entropy conditions A.2 and A.3 . Then, for $t \geq 0$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_1 \leq \|u(\cdot,0) - v(\cdot,0)\|_1 \tag{A.6}$$

Proof. Assume that $k_L < k_R$, the other case being similar. Following [20], the quantity on the left of A.6 is broken up into integrals over segments where $u - v$ does not change sign. This decomposition gives

$$\|u(\cdot,t) - v(\cdot,t)\|_1 = \sum_i \sigma_i \int_{x_i}^{x_{i+1}} (u(x,t) - v(x,t))dx \tag{A.7}$$

where $\sigma_i = \sigma(u - v)$, which is constant within the interval (x_i, x_{i+1}). Differentiating A.7 with respect to time gives terms of the form

$$\begin{aligned} \sigma_i \frac{d}{dt} \int_{x_i}^{x_{i+1}} (u(x,t) - v(x,t))dx = \sigma_i \Big[&\dot{x}(u(x) - v(x)) \\ &- (kf(u(x)) - kf(v(x))) \Big]_{x_i}^{x_{i+1}} \end{aligned} \tag{A.8}$$

In [20], it is noted that A.8 holds even if there are shocks in the interior of (x_i, x_{i+1}). For the same reason, i.e., the Rankine-Hugoniot condition, A.8 holds even if the jump in k lies within (x_i, x_{i+1}). For x_i (or x_{i+1}) away from the jump in k, the contribution to A.8 from x_i (or x_{i+1}) is nonpositive, since the argument in [20] applies in this case, due to the geometric entropy condition A.3. The only case remaining is where $u - v$ changes sign at the jump in k. In that case there are contributions to A.7 from two

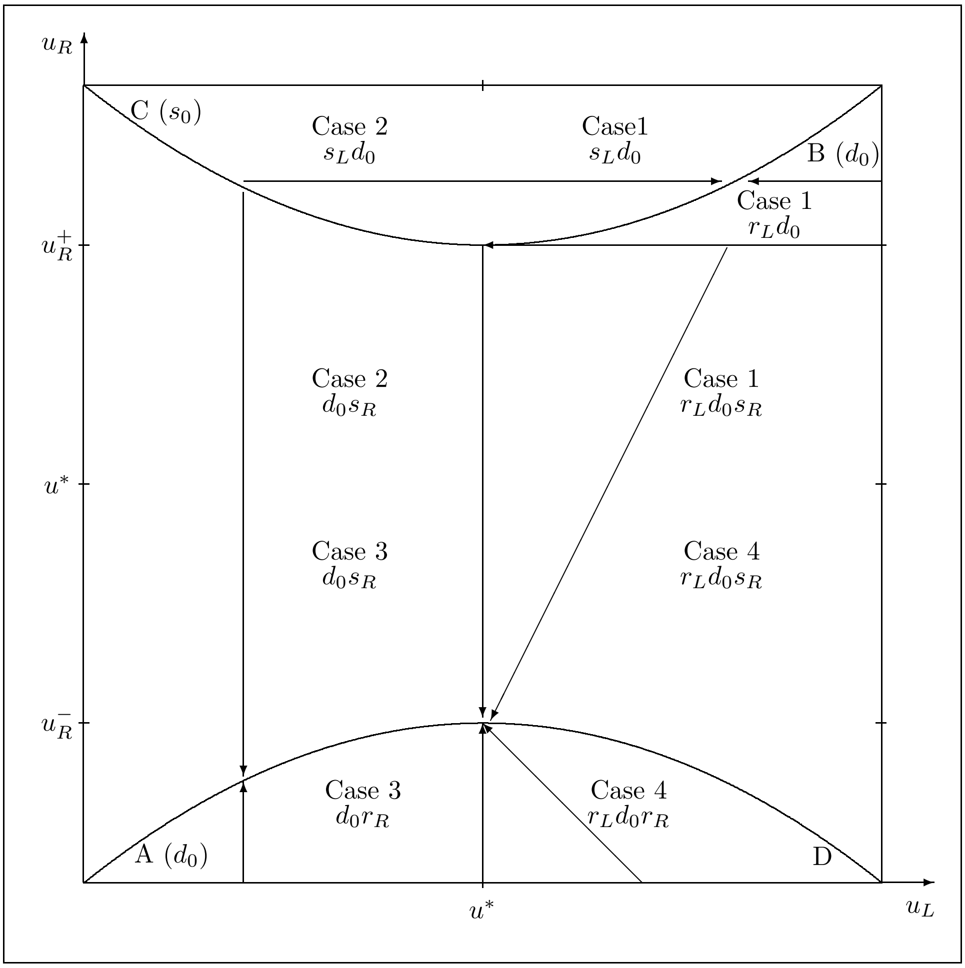

FIG. A.2. *The mapping* $(u_L, u_R) \mapsto (\tilde{u}_L, \tilde{u}_R)$ *for* $0 < k_L < k_R$. *The segments of the curves* $k_L f(u_L) = k_R f(u_R)$ *labeled A and B correspond to admissible (non-shock) discontinuities. The segments labeled C and D correspond to entropy-correct shocks,* s_0, *and entropy violating discontinuities, respectively. (See Figure A.1.) The states* u_R^- *and* u_R^+ *are the two solutions of* $k_R f(u) = k_L f(u^*)$.

consecutive terms of the form A.8. Taking into account that $\dot{x} = 0$ along the line $x = 0$, the total contribution is

$$\sigma_i \Big[-k_L(f(u(x)) - f(v(x))) \Big]_{0^-} + \sigma_{i+1} \Big[k_R(f(u(x)) - f(v(x))) \Big]_{0^+} \tag{A.9}$$

The situation where $\sigma_i = 0$ or $\sigma_{i+1} = 0$ can be eliminated, since then the Rankine-Hugoniot condition implies that both σ_i and σ_{i+1} vanish. Take the case where $u > v$ in (x_{i-1}, x_i), $u < v$ in (x_i, x_{i+1}), i.e., $\sigma_i = 1$, $\sigma_{i+1} = -1$. The other case is similar. Then A.9 becomes

$$k_L f(v_L) + k_R f(v_R) - k_L f(u_L) - k_R f(u_R) \tag{A.10}$$

In order for there to be a sign change, at least one of $u_R < u_L$, $v_R > v_L$ must hold. Take the case where $u_R < u_L$. Due to the geometric entropy condition A.2,

$u_R < u_L \Rightarrow u_L < u^*$. Since $v_L < u_L \leq u^*$, and f is increasing in $(0, u^*)$, it follows that $f(v_L) < f(u_L)$. Then A.10 becomes

$$k_L f(v_L) + k_R f(v_R) - k_L f(u_L) - k_R f(u_R) = 2k_L(f(v_L) - f(u_L)) < 0 \tag{A.11}$$

Next, take the case where $v_R > v_L$. It follows from the Rankine-Hugoniot condition, and the assumption that $k_L < k_R$, that $v_R > u^*$. If $v_L > u^*$, then $u_L > v_L \Rightarrow u_L > u^*$. The entropy condition A.2 then requires that $u_R > u^*$. But, then $u_L > v_L \Rightarrow u_R > v_R$, which is a contradiction, so it must be that $v_L < u^*$. If also $u_L < u^*$, then $v_L < u_L \Rightarrow f(v_L) < f(u_L)$. If $u_L > u^*$, the entropy condition A.2 requires that $u_R > u^*$. Then, $u_R < v_R \Rightarrow f(u_R) > f(v_R)$. Thus, in either case A.10 becomes

$$2k_L(f(v_L) - f(u_L)) = 2k_R(f(v_R) - f(u_R)) < 0 \tag{A.12}$$

□

The next two propositions are used in Section 4 to show that the limit u generated by the numerical scheme also satisfies the geometric entropy conditions.

PROPOSITION A.3. *Let $F(u) = \sigma(u - u^*)(f(u) - f(u^*))$. The right and left intermediate states of the solution of the Riemann problem constructed above satisfy*

$$k_R F(\tilde{u}_R) - k_L F(\tilde{u}_L) \leq |k_R - k_L| f(u^*) \tag{A.13}$$

For a pair of states (v_L, v_R) which satisfy the Rankine-Hugoniot condition A.1, but fail the geometric entropy condition A.2,

$$k_R F(v_R) - k_L F(v_L) > |k_R - k_L| f(u^*) \tag{A.14}$$

Proof. Take $k_L \leq k_R$; the other case is similar. By considering the various cases for a pair of states (v_L, v_R) which satisfy the Rankine-Hugoniot condition A.1, the following relationships result:

Case 1. $v_R > u^* > v_L \Rightarrow k_R F(v_R) - k_L F(v_L) = k_R(f_R - f^*) + k_L(f_L - f^*) \leq 0$
Case 2. $v_R, v_L \geq u^* \Rightarrow k_R F(v_R) - k_L F(v_L) = (k_L - k_R) f^*$
Case 3. $v_R, v_L \leq u^* \Rightarrow k_R F(v_R) - k_L F(v_L) = (k_R - k_L) f^*$
Case 4. $v_R < u^* < v_L \Rightarrow k_R F(v_R) - k_L F(v_L) = k_R(f^* - f_R) + k_L(f^* - f_L) > 0$

The first three cases cover the situation where the geometric entropy condition A.2 is satisfied, from which A.13 follows, by Proposition A.1. Inequality A.14 follows from Case 4., the only case where A.2 fails. In fact, since $v_L > u^* \Rightarrow f_L < f^*$,

$$\begin{aligned} k_R(f^* - f_R) + k_L(f^* - f_L) &= (k_R + k_L) f^* - 2k_L f_L \\ &> (k_R + k_L) f^* - 2k_L f^* = (k_R - k_L) f^* \end{aligned}$$

□

A slightly more extensive consideration of the various cases gives the following proposition.

PROPOSITION A.4. *For any real number c, let $F(u) = \sigma(u - c)(f(u) - f(c))$. The right and left intermediate states of the solution of the Riemann problem constructed above satisfy*

$$k_R F(\tilde{u}_R) - k_L F(\tilde{u}_L) \leq |k_R - k_L| f(c) \tag{A.15}$$

Proof. As before, assume that $k_L < k_R$, the other case being similar. If $\tilde{u}_L, \tilde{u_R} \leq c$ or $\tilde{u}_L, \tilde{u_R} \geq c$, the result follows as in Cases 2. and 3. of the proof of Proposition A.3. So, assume that c lies between $\tilde{u}_R$ and $\tilde{u}_R$. The rest of the proof involves the consideration of three cases:

Case A. $\tilde{u}_L, \tilde{u_R} \leq u^*$. Due to the fact that $k_L < k_R$, it must be that $\tilde{u}_L \geq c \geq \tilde{u_R}$. Then

$$\begin{aligned} k_R F_R - k_L F_L &= (k_R + k_L) f(c) - 2k_L f_L \\ &\leq (k_R + k_L) f(c) - 2k_L f(c) = (k_R - k_L) f(c) \end{aligned}$$

(A.16)

Here, the first inequality follows from $f_L \geq f(c)$, since f is increasing in the interval of interest.

Case B. $\tilde{u}_L, \tilde{u_R} \geq u^*$. Due to the fact that $k_L > k_R$, it must be that $\tilde{u}_R \geq c \geq \tilde{u_L}$. Then

$$\begin{aligned} k_R F_R - k_L F_L &= 2k_R f_R - (k_R + k_L) f(c) \\ &\leq 2k_R f(c) - (k_R + k_L) f(c) = (k_R - k_L) f(c) \end{aligned}$$

(A.17)

Case C. $\tilde{u}_L < u^* < \tilde{u_R}$. In this case,

$$\begin{aligned} k_R F_R + k_L F_L &= k_R (F_R - f(c)) + k_L (F_L - f(c)) \\ &\leq 0 \leq |k_R - k_L| f(c) \end{aligned}$$

(A.18)

□

The following additivity property is a simple consequence of the fact that the solution to the Riemann problem is a similarity solution, i.e., u is constant along the half lines $\xi = x/t$, $t > 0$.

PROPOSITION A.5. *Let $u(x,t)$ be the similarity solution constructed above to the Riemann problem with data (u_L, u_R) and (k_L, k_R). Let*

$$v(x,t) = \begin{cases} u(x,t) & , x \leq 0 \\ \tilde{u}_L & , x \geq 0 \end{cases}$$

$$w(x,t) = \begin{cases} \tilde{u}_R & , x \leq 0 \\ u(x,t) & , x \geq 0 \end{cases}$$

Then v is the unique entropy solution to the constant-k Riemann problem with data (k_L, k_L), $(u_L, \tilde{u}_L)$ and w is the unique entropy solution to the constant-k Riemann problem with data (k_R, k_R), $(\tilde{u}_R, u_R)$. Both v and w are continuous across the jump in k for $t > 0$.

Proof. Only the uniqueness claim needs to be addressed. Take $v(x,t)$, the situation for w being similar. In constructing v from u, the discontinuity at $x = 0$ has been removed, and there are no discontinuities in $x > 0$, since v is constant there. Thus, any discontinuites lie in $x < 0$, and all such discontinuities satisfy A.3, which implies uniqueness for piecewise smooth solutions [20]. □

The significance of Proposition A.5 is that $k_L f(\tilde{u}_L)$ and $k_R f(\tilde{u}_R)$ can be interpreted as constant-k Godunov fluxes

$$k_L f(\tilde{u}_L) = k_L h(\tilde{u}_L, u_L), \;\; k_R f(\tilde{u}_R) = k_R h(u_R, \tilde{u}_R) \tag{A.21}$$

This property is used to derive discrete entropy inequalities for the Godunov version of the difference scheme in Section 4.

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