CONVERGENCE OF A DIFFERENCE SCHEME FOR CONSERVATION LAWS WITH A DISCONTINUOUS FLUX

JOHN D. TOWERS *

Abstract. Convergence is established for a scalar finite difference scheme, based on the Godunov or Engquist-Osher flux, for scalar conservation laws having a flux that is spatially dependent through a possibly discontinuous coefficient. Other works in this direction have established convergence for methods employing the solution of 2x2 Riemann problems. The algorithm discussed here uses only scalar Riemann solvers. Satisfaction of a set of Kruzkov-type entropy inequalities is established for the limit solution, from which geometric entropy conditions follow. Assuming a piecewise constant coefficient, it is shown that these conditions imply L^1-contractiveness for piecewise C^1 solutions, thus extending a well known theorem.

Key words. conservation laws, difference approximations, discontinuous coefficients

AMS subject classifications. 35L65, 65M06, 65M12, 35R05

1. Introduction. The subject of this paper is a finite difference algorithm for computing approximate solutions of the Cauchy problem for scalar conservation laws of the form

$$u_t + (k(x)f(u))_x = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

where $(x,t) \in \mathbf{R} \times [0,\infty)$. The flux $k(x)f(u)$ has a possibly discontinuous spatial dependence through the coefficient k, which is allowed to have jump discontinuities. A simple physical model corresponding to 1.1 is the Witham model of car traffic flow on a highway [11]. The spatially varying coefficient k corresponds to changing road conditions. Equations of this type have been addressed by several authors in recent years, often as the simplest examples of nonstrictly hyperbolic systems [6], [7], [20]. This approach is based on the observation that 1.1 can be written as a 2x2 system:

$$u_t + (kf(u))_x = 0, \quad k_t = 0. \tag{1.2}$$

The eigenvalues associated with this system are $\lambda_u = kf'(u)$ and $\lambda_k = 0$. The system fails to be strictly hyperbolic if f' can vanish, in which case the system is described as resonant.

Even when k and the initial data u_0 are smooth, solutions to 1.1 develop discontinuities, and so weak solutions are sought. A weak solution is a bounded measurable function $u : \mathbf{R} \times [0,\infty) \to \mathbf{R}$ satisfying

$$\int_{\mathbf{R}\times\mathbf{R}^+} (\psi_t u + \psi_x kf(u))dxdt + \int_{\mathbf{R}} \psi(x,0)u_0(x)dx = 0 \tag{1.3}$$

for all $\psi \in C_0^\infty(\mathbf{R} \times [0,\infty))$.

This paper considers the conservation law 1.1 within the framework of scalar finite difference schemes. The spatial domain $\mathbf{R}$ is divided into cells $I_j = [x_j - \Delta x/2, x_j + \Delta x/2)$ with centers at the points $x_j = j\Delta x$ for $j \in \mathbf{Z}$. Similarly, the time domain $[0,\infty)$ is discretized via $t^n = n\Delta t$ for $n \in \{0,1,\ldots\}$, resulting in time strips $I^n = [t^n, t^{n+1})$. Let $\chi_{j,n}$ be the characteristic function for the rectangle $R_j^n = I_j \times I^n$.

*October 30, 1999 jtowers@cts.com

The finite difference scheme then generates, for each mesh size $\Delta = \Delta x$, a piecewise constant approximation

$$u^{\Delta}(x,t) = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \chi_{j,n}(x,t) u_j^n.$$

The initial data is discretized via the piecewise constant approximation

$$u^{\Delta}(x,0) = \sum_j \chi_j(x) u_j^0, \qquad u_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x)dx,$$

where $\chi_j(x)$ is the characteristic function for the interval I_j. The approximations u_j^n are generated by the explicit, conservation-form, time marching algorithm

$$u_j^{n+1} = u_j^n - \lambda(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}} h_{j-\frac{1}{2}}). \tag{1.4}$$

Here, $\lambda = \Delta t/\Delta x$, and $h_{j+\frac{1}{2}}$ is based on a two point monotone (nondecreasing with respect to the second argument, nonincreasing with respect to the first) numerical flux, $h(v,u)$, consistent with the actual flux, i.e., $h(u,u) = f(u)$. In order to maintain the monotonicity of the scheme, the arguments of the numerical flux are transposed when the coefficient k is negative, so that

$$h_{j+\frac{1}{2}} = \begin{cases} h(u_{j+1}, u_j), & \text{if } k_{j+\frac{1}{2}} \geq 0 \\ h(u_j, u_{j+1}), & \text{if } k_{j+\frac{1}{2}} < 0 \end{cases}. \tag{1.5}$$

In particular, this paper focuses on the closely-related Godunov [3] and Engquist-Osher (EO) [2] fluxes. The Godunov flux is derived by applying the exact solution operator for 1.1, with constant $k = 1$, to piecewise constant initial data. The numerical flux that results is then $h(v,u) = f(u^G(v,u))$. Here $u^G(v,u)$ is the similarity solution of the resulting Riemann problem with right and left states v and u, evaluated anywhere along the vertical half line $t > 0$ in (x,t)-space where the jump in the initial data occurs. The following formula for the Godunov flux is derived in [14]:

$$h(v,u) = \begin{cases} \min_{[u,v]} f(w), & \text{if } u \leq v \\ \max_{[v,u]} f(w), & \text{if } u \geq v \end{cases}. \tag{1.6}$$

The Godunov flux is Lipschitz continuous, and for $f \in C^1$ it has piecewise continuous partial derivatives. With the notational convention that $a_- = \min(a,0)$, $a_+ = \max(a,0)$, it follows from 1.6 that

$$f'_-(v) \leq h_v(v,u) \leq 0 \leq h_u(v,u) \leq f'_+(u). \tag{1.7}$$

The EO flux, which is defined by

$$h(v,u) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2} \int_u^v |f'(w)| dw, \tag{1.8}$$

is also Lipschitz continuous, but is smoother than the Godunov flux. For $f \in C^1$, it has continuous partial derivatives satisfying

$$f'_-(v) = h_v(v,u) \leq 0 \leq h_u(v,u) = f'_+(u). \tag{1.9}$$

It follows from 1.7 and 1.9 that $\|f'\|_\infty$ serves as a Lipschitz constant for both fluxes. In the case where the flux f is convex or concave, with a single stagnation point, the EO and Godunov fluxes are identical, except for the single case where the two arguments are joined by a sonic shock, i.e., $f'(v) < 0 < f'(u)$.

In the scheme 1.4, the coefficient k is approximated at each cell boundary, resulting in a discretized version of k,

$$k^\Delta(x) = \sum_j \chi_{j+\frac{1}{2}}(x) k_{j+\frac{1}{2}}, \qquad k_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k(x)dx,$$

where $\chi_{j+\frac{1}{2}}(x)$ is the characteristic function for the interval $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$. The discretized version k^Δ has jumps at cell centers, as opposed to cell boundaries, i.e., the discretization of k is staggered with respect to that of u. This results in a reduction in complexity, compared with the approach where the two discretizations are aligned. When the discretizations of the conserved quantity and flux coefficient are aligned, more complicated 2x2 Riemann problems arise, and thus a 2x2 Riemann solver is required. The concept of reducing complexity via mesh staggering is well known.

In [19], Temple used the Glimm scheme, solving 2x2 Riemann problems, to establish existence and uniqueness for a 2x2 resonant system of conservation laws. The singular function Ψ, used to prove compactness in §3 of this paper, originated in [19], in a slightly different form. (Unlike the system 1.2, the system studied was not reducible to a scalar conservation law, so the scalar scheme 1.4 discussed in this paper is not applicable to that problem.) The authors of [12] and [13] studied the 2x2 Godunov method as it applies to

$$u_t + f(k(x), u)_x = 0. \tag{1.10}$$

They adjoined the equation $k_t = 0$, and treated 1.10 as a 2x2 system, for the purpose of modeling a resonant system. Using Ψ to establish compactness, they proved that the 2x2 Godunov method is convergent (modulo extraction of a subsequence) to a weak solution. They established a bound on the total variation of a function that corresponds to the function z^Δ of this paper, but their method of analysis was different, entailing a study of the various waves arising from the 2x2 Riemann problems. In [13] it was observed that time independent bounds on derivatives (measured via Ψ) had been achieved for the 2x2 Glimm and Godunov schemes, but not for any scalar scheme that applies to 1.10. Section 3 of this paper provides a bound of this type for equations of the form 1.1, which constitute a nontrivial subset of 1.10, and are sometimes used to model the more general 1.10 [6], [7], [20] .

Outside of the difference scheme methods, the front tracking approach has been effective, computationally, and as an analytical tool for studying solutions to 1.1. Reference [7] establishes existence, uniqueness, and asymptotic behavior of the solution of 1.1, under essentially the same concavity assumptions about the flux f as this paper. The singular function Ψ, in the form appearing in §3 of this paper, is used in [7] also. A bound on the variation, measured via Ψ, is achieved by studying the wave interactions that arise. Additionally, satisfaction of a wave entropy condition is established for the limit of the front tracking scheme, and uniqueness is shown to follow from this entropy condition. In [6], it is shown that solutions of 1.1 and 1.10 depend continuously on k.

Reference [20] discusses computational difficulties that can arise with schemes for 1.1 and 1.10 that are based on the solution of 2x2 Riemann problems. Specifically, the

random choice method and front tracking method can fail, due to variation blow-up, and non-existence of solutions to 2x2 Riemann problems. The 2x2 Lax-Friedrichs scheme does not experience these failures, as is shown in [20] by examples, since it avoids solving Riemann problems. The scheme 1.4 discussed in this paper, which is based on scalar Riemann solvers, is also unaffected by these difficulties. In the example provided in [20] where a solution to the Riemann problem fails to exist, the source of this failure is a sign change in k. This provides the motivation for addressing the case of indefinite k in §1 and §2 of this paper.

This paper is organized as follows. Section 2 discusses the monotonicity and L^1-contractiveness properties of the scheme 1.4, including the case of indefinite k. Section 3 focuses on the case of positive k, and establishes the main result of the paper, convergence of a subsequence to a weak solution of the conservation law. Section 4 addresses the related issues of uniqueness and entropy satisfaction for the limit solution.

2. Monotonicity. For the case of constant k, the theory of monotone schemes has been essentially complete for many years [1], [5], [9], [17]. In that setting, approximations generated by monotone schemes are well-known to share many of the properties of the actual entropy weak solution to the conservation law 1.1, including monotonicity, L^1-contractiveness, and satisfaction of a discrete entropy inequality. Their major drawback is that they are at best only first order accurate even in regions where the solution is smooth [5]. Nevertheless, they provide the starting point for many of the modern higher order accurate schemes, some of which are constructed by modifying the two-point monotone flux function with higher order correction terms, and then applying flux limiters [4], [15]. The flux limiters damp out spurious oscillations that the correction terms often generate in regions of rapid transition.

A finite difference scheme such as 1.4 is monotone if

$$u_j^n \leq v_j^n, \quad \forall j \quad \Rightarrow \quad u_j^{n+1} \leq v_j^{n+1}, \quad \forall j.$$

When k is constant, monotonicity of the scheme follows from monotonicity of the numerical flux h under suitable CFL conditions. It follows that, in addition, the computed approximations u_j^n remain within the convex hull of the initial data for all n. The following proposition provides an analog of these properties for the variable k situation.

PROPOSITION 2.1. *With the CFL condition* $2\lambda\|k\|_\infty\|f'\|_\infty \leq 1$ *both the Godunov and EO versions of the scheme are monotone. If k is nonnegative or nonpositive the CFL condition can be relaxed to* $\lambda\|k\|_\infty\|f'\|_\infty \leq 1$. *If the initial data* $u_0(x)$ *lies within the interval* $\Omega_0 = [\underline{u}, \overline{u}]$, *and* $f(\underline{u}) = f(\overline{u}) = 0$, *then* $u_j^n \in \Omega_0$, *for all j, and all* $n \geq 0$.

Proof. Let $u_j = u_j^n$. Expressing the three-point scheme as $u_j^{n+1} = G(u_{j-1}, u_j, u_{j+1})$, the proof proceeds by showing that the partial derivatives $\partial G/\partial u_{j+i}$ are each nonnegative, from which monotonicity of the scheme follows. That the partial derivative of G with respect to u_{j-1} is nonnegative is clear from

$$\partial G/\partial u_{j-1} = \begin{cases} \lambda k_{j-\frac{1}{2}} h_u(u_j, u_{j-1}), & \text{if } k_{j-\frac{1}{2}} \geq 0 \\ \lambda k_{j-\frac{1}{2}} h_v(u_{j-1}, u_j), & \text{if } k_{j-\frac{1}{2}} \leq 0 \end{cases}.$$

A similar formula for $\partial G/\partial u_{j+1}$ shows that it is also nonnegative. For $\partial G/\partial u_j$, there are four cases, depending on the signs of $k_{j-\frac{1}{2}}$ and $k_{j+\frac{1}{2}}$. If $k_{j-\frac{1}{2}} \geq 0$ and $k_{j+\frac{1}{2}} \geq 0$,

$$\frac{\partial G}{\partial u_j} = 1 - \lambda k_{j+\frac{1}{2}} h_u(u_{j+1}, u_j) + \lambda k_{j-\frac{1}{2}} h_v(u_j, u_{j-1})$$

$$\geq 1 - \lambda k_{j+\frac{1}{2}} f'_+(u_j) + \lambda k_{j-\frac{1}{2}} f'_-(u_j)$$
$$\geq 1 - \lambda \|k\|_\infty f'_+(u_j) + \lambda \|k\|_\infty f'_-(u_j) \geq 1 - \lambda \|k\|_\infty \|f'\|_\infty \geq 0.$$

The case where $k_{j-\frac{1}{2}} \leq 0$ and $k_{j+\frac{1}{2}} \leq 0$ is similar, and the calculation is omitted. For the case where $k_{j-\frac{1}{2}} < 0$ and $k_{j+\frac{1}{2}} > 0$,

$$\begin{aligned}\frac{\partial G}{\partial u_j} &= 1 - \lambda k_{j+\frac{1}{2}} h_u(u_{j+1}, u_j) + \lambda k_{j-\frac{1}{2}} h_u(u_{j-1}, u_j)\\ &\geq 1 - \lambda k_{j+\frac{1}{2}} f'_+(u_j) + \lambda k_{j-\frac{1}{2}} f'_+(u_j)\\ &\geq 1 - 2\lambda \|k\|_\infty f'_+(u_j) \geq 1 - 2\lambda \|k\|_\infty \|f'\|_\infty \geq 0.\end{aligned}$$

The case where $k_{j-\frac{1}{2}} > 0$ and $k_{j+\frac{1}{2}} < 0$ is similar and is omitted. The stated invariance of Ω_0 with respect to the computed solution u_j^n follows from the monotonicity of the scheme, along with the fact that f vanishes at $\underline{u}$ and $\overline{u}$. Specifically,

$$\underline{u} = G(\underline{u}, \underline{u}, \underline{u}) \leq G(u_{j-1}, u_j, u_{j+1}) \leq G(\overline{u}, \overline{u}, \overline{u}) = \overline{u}.$$

□

For the remainder of this paper it will be assumed that $u_0 \in L^1 \bigcap L^\infty \bigcap BV$ and $k \in L^1_{loc} \bigcap L^\infty \bigcap BV$. Here BV denotes the space of locally integrable functions w having bounded total variation, denoted $TV(w)$. Also, it will be assumed that k is constant for large x, specifically, that there are constants $k(+\infty)$, $k(-\infty)$, and X such that $k(x) = k(-\infty)$ for $x \leq -X$, $k(x) = k(+\infty)$ for $x \geq +X$. With the cell-average type discretizations discussed in §1 for u_0 and k, the discretized versions also satisfy these assumptions, and the various norms of the discretized quantities, u_0^Δ and k^Δ, are bounded uniformly in Δ by the corresponding norms for u_0 and k [1].

PROPOSITION 2.2. *With the CFL conditions in Proposition 2.1 and the assumption that the initial data satisfies u_0, $v_0 \in L^1 \bigcap L^\infty$, $k \in L^\infty$, and k is constant for large x, the scheme 1.4 is L_1-contractive, i.e., the inequality*

$$\sum_j |v_j^{n+1} - u_j^{n+1}| \Delta x \leq \sum_j |v_j^n - u_j^n| \Delta x \tag{2.1}$$

holds for a pair of approximate solutions u_j^n and v_j^n generated by the scheme. The following inequality also holds:

$$\sum_j |u_j^{n+1} - u_j^n| \Delta x \leq \sum_j |u_j^1 - u_j^0| \Delta x. \tag{2.2}$$

Proof. Both solutions u_j^n and v_j^n remain in $L^1 \bigcap L^\infty$. The L^∞ bound follows from Proposition 2.1. The L^1 bound follows from the finite range of influence of the initial data, along with the assumption that k is constant for large x. These observations along with the fact that the scheme is monotone (due to the CFL condition) and conservative, add up to the hypotheses of the Crandall-Tartar Lemma [1], giving 2.1. The inequality 2.2 follows from L_1-contractiveness, applied inductively, to two successive time steps, u_j^n and u_j^{n+1}, of a single computed solution. □

3. Convergence. In the case where the coefficent k is constant, monotonicity implies that the scheme is Total Variation Decreasing (TVD). However, even for smooth, but nonconstant, k, the total variation of the solution to the conservation law generally increases, at least initially, and so there is no hope that the numerical scheme for variable k will be TVD.

As in [7], [12], and [19], the approach to convergence in this paper is to use the singular mapping Ψ first used by Temple in his study of a resonant hyperbolic system [19]. Some assumptions about the flux function $f(u)$ will be required before defining Ψ. The flux function $f \in C^2[0,1]$ is assumed to be strictly concave on $[0,1]$, i.e., $f''(u) \le \mu < 0$, and $f(0) = f(1) = 0$. Within the interval $(0,1)$, $f(u) > 0$, with a single maximum f^* at $u^* \in (0,1)$. These are similar to the conditions imposed on f in [6] and [7]. Then the singular function Ψ is defined by

$$\Psi(u,k) = k\sigma(u-u^*)(1 - f(u)/f^*) = \frac{k}{f^*}\int_{u^*}^{u} |f'(w)|dw.$$

Here $\sigma(w) = w/|w|$ if $w \ne 0$, and $\sigma(0) = 0$. In the remainder of this paper, k will be assumed to be bounded and strictly positive: $0 < \underline{k} \le k(x) \le \overline{k}$. Then 1.5 becomes $h_{j+\frac{1}{2}} = h(u_{j+1}, u_j)$. The convergence result of this paper remains valid (with a more restrictive CFL condition, and a larger bound on the variation of z^{Δ}) if k is allowed to have finitely many points x where $k(x^-)k(x^+) \le 0$. In order to avoid obscuring the main idea of the paper, the analysis of this more general case is not presented.

For each value of k in $[\underline{k}, \overline{k}]$, $\Psi(\cdot,k) : [0,1] \to [-k,k]$ is an increasing, $1-1$ mapping. It is regular everywhere, except at the stagnation point u^*, where both f' and $\partial\Psi/\partial u$ vanish. The following Lipschitz continuity relationships in u and k follow directly from the definition of Ψ and the conditions imposed on the flux f:

$$|\Psi(u,k_1) - \Psi(u,k_2)| \le |k_1 - k_2|, \tag{3.1}$$

$$|\Psi(u_1,k) - \Psi(u_2,k)| \le k\frac{\|f'\|_\infty}{f^*}|u_1 - u_2|. \tag{3.2}$$

The function Ψ maps a function $w(x,t)$ into a new function $z(x,t)$ via $z(x,t) = \Psi(w(x,t), k(x))$. The following elementary facts concerning z are readily verified.

PROPOSITION 3.1. *Let* $w : \mathbf{R} \times [0,\infty) \to [0,1]$, *and* $0 < \underline{k} \le k(x) \le \overline{k}$.

1. *For each* $t \ge 0$, $z(\cdot,t) \in L^\infty$, *and* $\|z(\cdot,t)\|_\infty \le \|k\|_\infty = \overline{k}$.
2. *For each* $t \ge 0$, $z(\cdot,t) \in L^1_{loc}(\mathbf{R})$, *and* $\int_{-B}^{+B} |z(x,t)|dx \le 2B\overline{k}$.

It is now possible to state the main result of this paper.

THEOREM 3.2. *Let* $k \in L^1_{loc} \bigcap BV \bigcap L^\infty$, *with* k *taking constant values for large* x, *and assume that* $0 < \underline{k} \le k(x) \le \overline{k}$. *Let* $f \in C^2[0,1]$, $f'' \le \mu < 0$, $f(0) = f(1) = 0$, $f > 0$ *in* $(0,1)$, *with a single maximum* $f^* = f(u^*)$ *at* $u^* \in (0,1)$. *Let* $u_0 \in BV \bigcap L^1 \bigcap L^\infty$. *Let the mesh size* $\Delta \to 0$ *with* λ *fixed and satisfying the CFL condition* $\lambda\|k\|_\infty\|f'\|_\infty \le 1$, *resulting in the sequence of approximations* u^Δ. *Then, there is a weak solution* u *of 1.1 and a subsequence* u^{Δ_i} *such that* $u^{\Delta_i} \to u$ *a.e. and in* $L^1_{loc}(\mathbf{R} \times [0,\infty))$.

The rest of this section carries out the analysis required to prove Theorem 3.2. The basic approach is standard as in [17] and [18], with the exception that the quantity of immediate interest is $z^\Delta = \Psi(u^\Delta, k^\Delta)$ instead of the conserved variable u^Δ. Compactness for z^Δ follows in the usual way from bounds on the total variation and the L^∞ norm, along with a time-continuity estimate. Once the existence of a subsequential limit z has been established, the invertibility of Ψ then allows the corresponding

weak solution u to be recovered from the limit z, with $u^{\Delta} \to u$ guaranteed by the continuity of Ψ. Finally, a version of the Lax-Wendroff Theorem [10] demonstrates that the limit u is a weak solution to 1.1.

LEMMA 3.3. *Let $\phi(u) = \sigma(u - u^*)(f^* - f(u)) = f^*\Psi(u,k)/k$ and $\phi_j = \phi(u_j)$. For both the Godunov and EO versions of the scheme,*

$$(\phi_j - \phi_{j+1})_+ \le \chi_-(u_j)|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| + \chi_+(u_{j+1})|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}|.$$

where $\chi_+(u) = 1$ if $f'(u) > 0$, and 0 elsewhere; $\chi_-(u) = 1$ if $f'(u) < 0$ and 0 elsewhere.

Proof. Like $\Psi(u,k)$, ϕ is a strictly increasing function of u, so $(\phi_j - \phi_{j+1})_+ > 0$ iff $u_j > u_{j+1}$. So assume that $u_j > u_{j+1}$. For concave f, and $u_j \ge u_{j+1}$, the Godunov and EO fluxes are identical, so 1.9 holds for both fluxes. Also, for either flux,

$$h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = \int_{u_{j-1}}^{u_j} h_u(u_j,w)dw + \int_{u_j}^{u_{j+1}} h_v(w,u_j)dw.$$

Using the following decomposition of $\phi_j - \phi_{j+1}$,

$$\phi_j - \phi_{j+1} = \int_{u_{j+1}}^{u_j} |f'(w)|dw = \int_{u_{j+1}}^{u_j} f'_+(w)dw - \int_{u_{j+1}}^{u_j} f'_-(w)dw,$$

it suffices to show that the following two inequalities are satisfied:

$$\chi_-(u_j)|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| \ge \int_{u_j}^{u_{j+1}} f'_-(w)dw, \tag{3.3}$$

$$\chi_+(u_{j+1})|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}| \ge -\int_{u_j}^{u_{j+1}} f'_+(w)dw. \tag{3.4}$$

For 3.3, if $\chi_-(u_j) = 0$ then both u_j and u_{j+1} are to the left of u^*, so the integral vanishes, and the inequality holds. So assume that $\chi_-(u_j) = 1$, i.e., $u_j > u^*$. Since $u_{j+1} \le u_j$, $\int_{u_j}^{u_{j+1}} h_v(w,u_j)dw \ge 0$. If $u_{j-1} \ge u^*$, $\int_{u_{j-1}}^{u_j} h_u(u_j,w)dw = 0$, while if $u_{j-1} < u^*$, $\int_{u_{j-1}}^{u_j} h_u(u_j,w)dw \ge 0$. In either case 3.3 holds, since then

$$\begin{aligned} |h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| &= \int_{u_{j-1}}^{u_j} h_u(u_j,w)dw + \int_{u_j}^{u_{j+1}} h_v(w,u_j)dw \\ &\ge \int_{u_j}^{u_{j+1}} h_v(w,u_j)dw = \int_{u_j}^{u_{j+1}} f'_-(w)dw. \end{aligned}$$

For 3.4, if $\chi_+(u_{j+1}) = 0$, the integral vanishes. If $\chi_+(u_{j+1}) = 1$,

$$\begin{aligned} |h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}| &= -\int_{u_j}^{u_{j+1}} h_u(u_{j+1},w)dw - \int_{u_{j+1}}^{u_{j+2}} h_v(w,u_{j+1})dw \\ &\ge -\int_{u_j}^{u_{j+1}} h_u(u_{j+1},w)dw = -\int_{u_j}^{u_{j+1}} f'_+(w)dw. \end{aligned}$$

$\square$

Let Δ_+ and Δ_- denote forward and backward difference operators. For example, $\Delta_+ h_{j-\frac{1}{2}} = \Delta_- h_{j+\frac{1}{2}} = h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$.

THEOREM 3.4. *Let $z^n(x) = \Psi(u^\Delta(x,t^n), k^\Delta(x))$. With the assumptions stated in Theorem 3.2, $TV(z^n)$ is bounded uniformly, for all $n \geq 0$, and all $\Delta > 0$. Specifically,*

$$TV(z^n) \leq k(-\infty) - k(+\infty) + 6TV(k) + \frac{4\overline{k}}{f^*}\|f'\|_\infty TV(u_0). \tag{3.5}$$

Proof. The superscript n is suppressed in the proof, except where two time levels are of interest. Taking into account the jumps in z at the cell boundaries (due to jumps in u^Δ) and the jumps at the cell centers (due to jumps in k^Δ), the total variation of $z(x)$ is

$$TV(z) = \sum_j |\Delta_+^u z_j| + \sum_j |\Delta_+^k z_{j-\frac{1}{2}}|, \tag{3.6}$$

where $\Delta_+^u z_j = \Psi(u_{j+1}, k_{j+\frac{1}{2}}) - \Psi(u_j, k_{j+\frac{1}{2}})$ and $\Delta_+^k z_{j-\frac{1}{2}} = \Psi(u_j, k_{j+\frac{1}{2}}) - \Psi(u_j, k_{j-\frac{1}{2}})$. The second sum in 3.6, due to jumps in k^Δ, is bounded by $TV(k)$, using 3.1. For the first sum, let $\kappa = k(-\infty) - k(+\infty)$. Summing over all of the jumps, and using the fact that $z(x) \to -k(\pm\infty)$ as $x \to \pm\infty$, results in

$$\sum_j \Delta_+^u z_j + \sum_j \Delta_+^k z_{j-\frac{1}{2}} = \kappa. \tag{3.7}$$

It follows from 3.7 that

$$\sum_j (\Delta_+^u z_j)_+ = \kappa - \sum_j (\Delta_+^u z_j)_- - \sum_j \Delta_+^k z_{j-\frac{1}{2}} \leq \kappa - \sum_j (\Delta_+^u z_j)_- + TV(k).$$

Then,

$$\sum_j |\Delta_+^u z_j| = \sum_j (\Delta_+^u z_j)_+ - \sum_j (\Delta_+^u z_j)_- \leq \kappa - 2\sum_j (\Delta_+^u z_j)_- + TV(k).$$

The following identity will be useful in estimating $-\sum_j (\Delta_+^u z_j)_-$:

$$\begin{aligned} -\sum_j (\Delta_+^u z_j)_- &= \sum_j (\Psi(u_j, k_{j+\frac{1}{2}}) - \Psi(u_{j+1}, k_{j+\frac{1}{2}}))_+ \\ &= \frac{1}{f^*} \sum_j k_{j+\frac{1}{2}} (\phi_j - \phi_{j+1})_+. \end{aligned} \tag{3.8}$$

By Lemma 3.3,

$$\begin{aligned} k_{j+\frac{1}{2}}(\phi_j - \phi_{j+1})_+ &\leq k_{j+\frac{1}{2}}(\chi_-(u_j)|\Delta_+ h_{j-\frac{1}{2}}| + \chi_+(u_{j+1})|\Delta_+ h_{j+\frac{1}{2}}|) \\ &\leq \chi_-(u_j)\big(|\Delta_+(k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})| + |h_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}}|\big) \\ &+ \chi_+(u_{j+1})\big(|\Delta_+(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}})| + |h_{j+\frac{3}{2}} \Delta_+ k_{j+\frac{1}{2}}|\big) \\ &\leq \chi_-(u_j)\big(|\Delta_+(k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})| + f^*|\Delta_+ k_{j-\frac{1}{2}}|\big) \\ &+ \chi_+(u_{j+1})\big(|\Delta_+(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}})| + f^*|\Delta_+ k_{j+\frac{1}{2}}|\big). \end{aligned}$$

The last inequality uses the fact that $|h_{j+\frac{1}{2}}| \leq f^*$, which follows directly from the conditions on f, along with 1.6 and 1.8. Substituting this into 3.8, shifting indices, and applying L^1-contractiveness gives

$$
\begin{aligned}
-\sum_j (\Delta_+^u z_j)_- &\leq \frac{1}{f^*}\sum_j ((\chi_+(u_j)+\chi_-(u_j))|\Delta_+(k_{j-\frac{1}{2}}h_{j-\frac{1}{2}})| + TV(k) \\
&\leq \frac{1}{\lambda f^*}\sum_j |u_j^{n+1}-u_j^n| + TV(k) \leq \frac{1}{\lambda f^*}\sum_j |u_j^1 - u_j^0| + TV(k).
\end{aligned}
\tag{3.9}
$$

The final step in the proof is to estimate the term $\sum_j |u_j^1 - u_j^0|$. For the jth term in this sum,

$$
\begin{aligned}
|u_j^1 - u_j^0| &\leq \lambda k_{j+\frac{1}{2}}|h(u_{j+1}^0, u_j^0) - h(u_j^0, u_{j-1}^0)| + \lambda|h(u_j^0, u_{j-1}^0)||\Delta_+ k_{j-\frac{1}{2}}| \\
&\leq \lambda k_{j+\frac{1}{2}}\|f'\|_\infty(|u_{j+1}^0 - u_j^0| + |u_j^0 - u_{j-1}^0|) + \lambda f^*|\Delta_+ k_{j-\frac{1}{2}}|.
\end{aligned}
$$

Summing over j results in

$$
\sum_j |u_j^1 - u_j^0| \leq 2\lambda\overline{k}\|f'\|_\infty TV(u_0) + \lambda f^* TV(k). \tag{3.10}
$$

Substituting this into the last inequality in 3.9 yields

$$
-\sum_j (\Delta_+^u z_j)_- \leq \frac{2\overline{k}}{f^*}\|f'\|_\infty TV(u_0) + 2TV(k).
$$

When all of the terms are collected, the bound 3.5 stated in the theorem results. □

Lemma 3.5. *With the assumptions stated in Theorem 3.2, there is a constant L, independent of the mesh size Δ, such that for $n > m \geq 0$,*

$$
\int_{\mathbf{R}} |z^\Delta(x,t^n) - z^\Delta(x,t^m)|dx \leq (n-m)L\Delta t. \tag{3.11}
$$

Proof. Taking into account the constant values of z^Δ in each half-cell, the integral on the left side of 3.11 is

$$
\frac{\Delta x}{2}\sum_j |\Psi(u_j^n, k_{j-\frac{1}{2}}) - \Psi(u_j^m, k_{j-\frac{1}{2}})| + \frac{\Delta x}{2}\sum_j |\Psi(u_j^n, k_{j+\frac{1}{2}}) - \Psi(u_j^m, k_{j+\frac{1}{2}})|.
$$

An application of 3.2 and L^1-contractiveness yields

$$
\begin{aligned}
\int_{\mathbf{R}} |z^\Delta(x,t^n) - z^\Delta(x,t^m)|dx &\leq \Delta x\overline{k}\frac{\|f'\|_\infty}{f^*}\sum_j |u_j^n - u_j^m| \\
&\leq \frac{\Delta t}{\lambda}\overline{k}\frac{\|f'\|_\infty}{f^*}(n-m)\sum_j |u_j^1 - u_j^0|.
\end{aligned}
$$

The proof is completed by bounding $\sum_j |u_j^1 - u_j^0|$ independently of Δ, using 3.10. $\square$

The following is essentially the Lax-Wendroff theorem [10].

THEOREM 3.6. *Let $u^\Delta : \mathbf{R}\times[0,\infty) \to [0,1]$ be a sequence of approximations computed via the scheme 1.4 which converges in $L^1_{loc}(\mathbf{R}\times[0,\infty))$ and with $TV(z^n)$ uniformly bounded, to $u \in L^1_{loc}(\mathbf{R}\times[0,\infty)) \bigcap L^\infty(\mathbf{R}\times[0,\infty))$. Then u is a weak solution of 1.1.*

Proof. Let $\psi \in C_0^\infty(\mathbf{R}\times[0,\infty))$ and $\Delta^t\psi_j^n = \psi_j^{n+1} - \psi_j^n$, $\Delta^x\psi_j^n = \psi_{j+1}^n - \psi_j^n$. Take $0 \le T \le N\Delta t$ such that $\psi = 0$ for $t \ge T$, and $0 \le B \le J\Delta x$ such that $\psi = 0$ for $|x| \ge B$. As in [10], multiply the scheme 1.4 by $\psi(x_j,t^n) = \psi_j^n$, sum over all j and $n \ge 0$, then sum by parts to get

$$\Delta t \sum_j \sum_n (u_j^n(\Delta^t\psi_j^n/\Delta t) + k_{j+\frac12}h_{j+\frac12}(\Delta^x\psi_j^n/\Delta x)) + \sum_j \psi_j^0 u_j^0 = 0,$$

where $h_{j+\frac12} = h(u^\Delta(x_{j+1},t^n), u^\Delta(x_j,t^n))$. By standard arguments [11], the sums involving $u_j^n(\Delta^t\psi_j^n/\Delta t)$ and $\psi_j^0u_j^0$ converge to their integral counterparts in 1.3, as $\Delta \to 0$. It remains to show that

$$\Delta x\Delta t \sum_j \sum_n k_{j+\frac12}h_{j+\frac12}(\Delta^x\psi_j^n/\Delta x) \to \int_{\mathbf{R}\times\mathbf{R}^+} kf(u)\psi_x dxdt. \tag{3.12}$$

The proof of 3.12 reduces to verifying that

$$\sum_j \int_{I_j} |k(x) - k_{j+\frac12}|dx \to 0, \quad \text{and} \tag{3.13}$$

$$\Delta t\Delta x \sum_{n=0}^{N} \sum_{|j|\le J} k_{j+\frac12} \int_{R_j^n} |h_{j+\frac12} - f(u)|dxdt \to 0. \tag{3.14}$$

The sum appearing in 3.13 does not exceed

$$\int_{\mathbf{R}} |k(x) - k^\Delta(x)|dx + \frac{\Delta x}{2}\int_{\mathbf{R}} \frac{|k(x+\Delta x/2) - k(x)|}{\Delta x/2}dx,$$

which proves 3.13. The expression on the left side of 3.14 is bounded by

$$\bar{k}\|f'\|_\infty \int_0^T \int_{-B}^{B} |u^\Delta - u|dxdt + \Delta t\Delta x \sum_{n=0}^{N}\sum_{|j|\le J} k_{j+\frac12}\left|\int_{u_j^n}^{u_{j+1}^n} |f'(w)|dw\right|. \tag{3.15}$$

The first integral in 3.15 tends to zero by assumption, and the following estimate

$$\begin{aligned}\Delta t\Delta x \sum_{n=0}^{N}\sum_{|j|\le J} k_{j+\frac12}\left|\int_{u_j^n}^{u_{j+1}^n} |f'(w)|dw\right| &\le \Delta t\Delta x \sum_{n=0}^{N}\sum_{|j|\le J} f^*|\Delta_+^u z_j^n| \\ &\le Tf^*\Delta x \max\{TV(z^n) | n\ge 0, \Delta > 0\}\end{aligned} \tag{3.16}$$

shows that the second integral also approaches zero. □

It is now possible to prove Theorem 3.2.

Proof. The CFL condition guarantees that the computed solutions u^Δ remain within $[0,1]$. An application of Proposition 3.1 gives uniform (with respect to both t and Δ) bounds on $\|z^\Delta(\cdot,t)\|_\infty$ and $\|z^\Delta(\cdot,t)\|_{L^1[-B,B]}$, for any compact interval $[-B,B]$. Theorem 3.4 provides a uniform bound on $TV(z^\Delta(\cdot,t))$. Finally, from Lemma 3.5, it follows that

$$\|z^\Delta(\cdot,t+\tau)-z^\Delta(\cdot,t)\|_1 \le C(|\tau|+\Delta),$$

where the constant C is independent of Δ and t. By standard compactness arguments applied to the sequence z^Δ, there is a subsequence, z^{Δ_i}, which converges in $L^1_{loc}(\mathbf{R}\times[0,\infty))$ to some function $z\in L^1_{loc}(\mathbf{R}\times[0,\infty))\bigcap L^\infty(\mathbf{R}\times[0,\infty))$. There is a further subsequence, also denoted z^{Δ_i}, which converges a.e. to z. Let $u(x,t)=\Psi^{-1}(z(x,t),k(x))$. Due to the strict monotonicity of $\Psi(\cdot,k)$, the function u is well-defined a.e., $u\in[0,1]$ a.e., and $u\in L^1_{loc}(\mathbf{R}\times[0,\infty))\bigcap L^\infty(\mathbf{R}\times[0,\infty))$. Dropping the subscript on Δ, it remains to show that $u^\Delta\to u$ a.e. and in $L^1_{loc}(\mathbf{R}\times[0,\infty))$. Let $B>0$, $T>0$. Using the fact that $u^\Delta=\Psi^{-1}(z^\Delta,k^\Delta)$ gives

$$\begin{aligned}\int_0^T\int_{-B}^B|u-u^\Delta|dxdt &\le \int_0^T\int_{-B}^B|\Psi^{-1}(z,k)-\Psi^{-1}(z^\Delta,k)|dxdt\\ &+\int_0^T\int_{-B}^B|\Psi^{-1}(z^\Delta,k)-\Psi^{-1}(z^\Delta,k^\Delta)|dxdt.\end{aligned}$$

The first integral tends to zero by the bounded convergence theorem, due to the continuity of Ψ^{-1} as a function of its first argument. For the second integral, an estimate of $|\Psi^{-1}(z^\Delta,k)-\Psi^{-1}(z^\Delta,k^\Delta)|$ is required. Implicit differentiation gives

$$\frac{\partial\Psi^{-1}}{\partial k}=\frac{\partial u}{\partial k}=-\Psi_k/\Psi_u=\frac{-1}{k}\frac{\int_{u^*}^u|f'(w)|dw}{|f'(u)|}.$$

When the numerator and denominator are each expanded in a Taylor series about u^*, the result is

$$\Big|\frac{\partial\Psi^{-1}}{\partial k}\Big|\le\frac{1}{2\underline{k}}\frac{\|f''\|_\infty}{|\mu|}|u-u^*|.$$

With this estimate, convergence of the second integral follows from the fact that $k^\Delta\to k$ in $L^1[-B,B]$. Having established convergence in $L^1_{loc}(\mathbf{R}\times[0,\infty))$, there is yet a further subsequence u^Δ, which converges to u a.e., as well as in $L^1_{loc}(\mathbf{R}\times[0,\infty))$. Theorem 3.6 proves that u is a weak solution of the conservation law. □

4. Entropy satisfaction. Even for constant k, solutions of 1.1 are not necessarily unique. Additional conditions, usually referred to as entropy conditions, are required to single out the physically relevant solution. When $k\in C^1$ the Kruzkov entropy condition applies [8]. Specifically, uniqueness is guaranteed if

$$\int_{\mathbf{R}\times\mathbf{R}^+}\big(|u-c|\psi_t+k\sigma(u-c)(f(u)-f(c))\psi_x-\sigma(u-c)k'f(c)\psi\big)dxdt\ge 0 \tag{4.1}$$

holds for all $c\in\mathbf{R}$, and for all nonnegative $\psi\in C_0^\infty(\mathbf{R}\times\mathbf{R}^+)$. The Kruzkov entropy condition 4.1 is not directly applicable if k is discontinuous, as was observed in [7].

For k discontinuous, the authors of [7] proved uniqueness within the class of solutions that satisfy a wave entropy condition. The wave entropy approach has the advantage of not requiring that the solution satisfy additional regularity conditions.

The approach in this section is to concentrate on the Godunov version of the scheme, and assume, as in [6] and [7], that k has finitely many jumps. Proposition 4.1 ensures that the limit solution u satisfies the Kruzkov entropy inequalities 4.1 locally, away from the jumps in k. Proposition 4.3 provides Kruzkov-type entropy inequalities that apply when the test function has support which intersects one or more jumps in k. Theorem 4.5 shows that these entropy inequalities imply geometric entropy conditions for piecewise C^1 solutions, and Theorem 4.6 applies the geometric entropy conditions to show uniqueness with the additional assumption that k is piecewise constant.

PROPOSITION 4.1. *In addition to the conditions stated in Theorem 3.2, let k be piecewise C^1, with a bounded derivative, $|k'(x)| \leq \beta$ for all x, and with finitely many jumps (in k and k') , located at $\xi_1 < \xi_2 < \ldots < \xi_M$. Let u^Δ be a convergent subsequence generated by the scheme 1.4 using the Godunov flux, converging to u, as in Theorem 3.2. Then the Kruzkov entropy inequalities 4.1 hold for every real number c, and every smooth test function $\psi \geq 0$ with compact support in $t > 0$, $x \in \mathbf{R} - \{\xi_1, \xi_2, \ldots, \xi_M\}$.*

To avoid complications arising from the discontinuity in $V'(u) = \sigma(u - c)$, the following entropy inequality is established for smooth V before proceeding with the proof of Proposition 4.1.

LEMMA 4.2. *In addition to the assumptions of Proposition 4.1, let (V, F) be a convex entropy pair for 1.1, i.e., V is convex and $F' = V'f'$, and assume that $V \in C^2[0,1]$. For every smooth test function $\psi \geq 0$ with compact support in $t > 0$, $x \in \mathbf{R} - \{\xi_1, \xi_2, \ldots, \xi_M\}$, and every $c \in \mathbf{R}$, the following inequality holds:*

$$\int_{\mathbf{R}\times\mathbf{R}^+} (V(u)\psi_t + kF(u)\psi_x)dxdt - \int_{\mathbf{R}\times\mathbf{R}^+} k'(V'(u)f(u) - F(u))\psi dxdt \geq 0.$$

Proof. Write the scheme 1.4 as

$$u_j^{n+1} = w_j^n - \lambda h_{j-\frac{1}{2}}\Delta_+ k_{j-\frac{1}{2}}, \text{ where } w_j^n = u_j^n - \lambda k_{j+\frac{1}{2}}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}).$$

Let $u^G_{j+\frac{1}{2}} = u^G(u^n_{j+1}, u^n_j)$. With the Godunov discrete entropy flux defined by $H_{j+\frac{1}{2}} = F(u^G_{j+\frac{1}{2}})$, the discrete entropy inequality

$$V(w_j^n) \leq V(u_j^n) - \lambda(k_{j+\frac{1}{2}}H_{j+\frac{1}{2}} - k_{j+\frac{1}{2}}H_{j-\frac{1}{2}})$$

holds for w_j^n. After rearranging terms, a discrete entropy inequality for the scheme 1.4 results:

$$V_j^{n+1} \leq V_j^n - \lambda\Delta_+(k_{j-\frac{1}{2}}H_{j-\frac{1}{2}}) + (V_j^{n+1} - V(w_j^n)) + \lambda H_{j-\frac{1}{2}}\Delta_+ k_{j-\frac{1}{2}}.$$

Multiplying by a smooth, nonnegative test function ψ with compact support in $t > 0$, $x \in \mathbf{R} - \{\xi_1, \xi_2, \ldots, \xi_M\}$, and proceeding as in the proof of Theorem 3.6 gives

$$\begin{aligned} &- \Delta x\Delta t \sum_{j,n} V_j^n(\Delta^t\psi_j^n/\Delta t) - \Delta x\Delta t \sum_{j,n} k_{j+\frac{1}{2}}H_{j+\frac{1}{2}}(\Delta^x\psi_j^n/\Delta x) \\ &- \Delta x\Delta t \sum_{j,n} \psi_j^n\Big(H_{j-\frac{1}{2}}(\Delta_+ k_{j-\frac{1}{2}}/\Delta x) + (V_j^{n+1} - V(w_j^n))/\Delta t\Big) \leq 0. \end{aligned} \tag{4.2}$$

As Δ approaches zero, the first sum in the top line of 4.2 converges to $\int\int V(u)\psi_t$, as in the proof of the Lax-Wendroff theorem. For the second sum in the first line of 4.2, convergence to $\int\int kF(u)\psi_x$ reduces to establishing

$$\Delta t \Delta x \sum_{n=0}^{N} \sum_{|j|\leq J} k_{j+\frac{1}{2}} \int_{R_j^n} |H_{j+\frac{1}{2}} - F(u)| dx dt \to 0, \tag{4.3}$$

where J, N, B, and T, are defined as in the proof of Theorem 3.6. Using the fact that $\|V'\|_\infty \|f'\|_\infty$ is a Lipschitz constant for $H_{j+\frac{1}{2}}$, the expression on the left side of 4.3 is bounded by

$$\bar{k}\|V'\|_\infty \|f'\|_\infty \int_0^T \int_{-B}^{B} |u^\Delta - u| dx dt + \Delta t \Delta x \sum_{n=0}^{N} \sum_{|j|\leq J} k_{j+\frac{1}{2}} \Big| \int_{u_j^n}^{u_{j+\frac{1}{2}}^G} |F'(w)| dw \Big|. \tag{4.4}$$

The first term in 4.4 tends to zero. Using $|F'| = |V'||f'|$ and the fact that $u_{j+\frac{1}{2}}^G$ lies between u_j^n and u_{j+1}^n, the second term does not exceed

$$\Delta t \Delta x \|V'\|_\infty \sum_{n=0}^{N} \sum_{|j|\leq J} k_{j+\frac{1}{2}} \Big| \int_{u_j^n}^{u_{j+1}^n} |f'(w)| dw \Big|.$$

The estimate 3.16 proves that this term tends to zero also. Consider the second line of 4.2. Since $k \in C^1$ within the support of ψ,

$$\Delta x \Delta t \sum_{j,n} \psi_j^n H_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}}/\Delta x) \to \int_{\mathbf{R}\times\mathbf{R}^+} \psi F(u) k' dx dt.$$

The last term in 4.2 to be dealt with is $\Delta x \Delta t \sum_{j,n} \psi_j^n (V_j^{n+1} - V(w_j^n))/\Delta t$. Expanding the divided difference in a Taylor series yields

$$\begin{aligned}(V_j^{n+1} - V(w_j^n))/\Delta t = &- V'(u_j^n) h_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}}/\Delta x) \\ &- \frac{1}{2} V''(\theta_1)(\lambda h_{j-\frac{1}{2}} \Delta_+ k_{j-\frac{1}{2}})^2/\Delta t \\ &- V''(\theta_2) h_{j-\frac{1}{2}} (\Delta_+ k_{j-\frac{1}{2}}/\Delta x)(u_j^{n+1} - u_j^n).\end{aligned} \tag{4.5}$$

Due to L^1-contractiveness, and the time continuity estimate 3.10, the sums involving the last two terms on the right side of 4.5 approach zero as $\Delta \to 0$. The first term on the right side of 4.5 approaches $-V'(u)f(u)k'$ in $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)$, which gives

$$\Delta x \Delta t \sum_{j,n} \psi_j^n (V_j^{n+1} - V(w_j^n))/\Delta t \to -\int_{\mathbf{R}\times\mathbf{R}^+} \psi V'(u) f(u) k' dx dt.$$

□

It is now possible to prove Proposition 4.1.

Proof. As in [8], approximate the convex function $|u-c|$ by a sequence of twice continuously differentiable convex functions V_i. Apply Lemma 4.2 for each V_i, and let $V_i \to |u-c|$. □

Let $\delta(x-\xi_i)$ denote the Dirac measure with support located at ξ_i.

PROPOSITION 4.3. *With the same assumptions as in Proposition 4.1, the following inequality holds for the special case where $c = u^*$, for all nonnegative $\psi \in C_0^\infty(\mathbf{R}\times\mathbf{R}^+)$:*

$$\int_{\mathbf{R}\times\mathbf{R}^+} \left(|u-u^*|\psi_t + k\sigma(u-u^*)(f(u)-f(u^*))\psi_x\right)dxdt$$
$$+f(u^*)\int_{\mathbf{R}\times\mathbf{R}^+}\Big(|k'(x)| + \sum_{i=1}^{M}|k(\xi_i^+)-k(\xi_i^-)|\delta(x-\xi_i)\Big)\psi dxdt \geq 0. \tag{4.6}$$

Proof. Proceeding as in the proof of Lemma 4.2 gives an inequality of the form 4.2, with $V(u) = |u-u^*|$, and the corresponding versions of $F(u)$ and $H_{j-\frac{1}{2}}$. The sums in the first line of 4.2 converge to their integral counterparts, as in the proof of Lemma 4.2. For the second line of 4.2, there are $\theta_j^n \in [-1,1]$ such that

$$H_{j-\frac{1}{2}}\frac{\Delta_+ k_{j-\frac{1}{2}}}{\Delta x} + \frac{V(u_j^{n+1})-V(w_j^n)}{\Delta t}$$
$$= \Big(\sigma(u^G_{j-\frac{1}{2}}-u^*)(f(u^G_{j-\frac{1}{2}})-f(u^*)) + \theta_j^n h_{j-\frac{1}{2}}\Big)(\Delta_+ k_{j-\frac{1}{2}}/\Delta x).$$

Using the fact that $h_{j-\frac{1}{2}} = f(u^G_{j-\frac{1}{2}}) \leq f(u^*)$ gives

$$|\sigma(u^G_{j-\frac{1}{2}}-u^*)(f(u^G_{j-\frac{1}{2}})-f(u^*)) + \theta_j^n h_{j-\frac{1}{2}}| \leq |f(u^*)|.$$

With this inequality, 4.2 becomes

$$-\Delta x\Delta t\sum_{j,n} V_j^n(\Delta^t\psi_j^n/\Delta t) - \Delta x\Delta t\sum_{j,n} k_{j+\frac{1}{2}}H_{j+\frac{1}{2}}(\Delta^x\psi_j^n/\Delta x)$$
$$-\Delta x\Delta t\sum_{j,n}\psi_j^n f(u^*)|\Delta_+ k_{j-\frac{1}{2}}/\Delta x| \leq 0. \tag{4.7}$$

The term in the bottom line of 4.7 converges to the integral in the second line of 4.6. This can be verified by breaking the spatial portion of the sum in the bottom line of 4.7 into sums over intervals where k is differentiable, and isolating the finite number of cells where the jumps in k are concentrated. ∎

Let u be a piecewise smooth solution to 1.1. It follows from a standard test function calculation that the Rankine-Hugoniot condition across a jump in k at one of the points $x = \xi_i$ is

$$k_L f(u_L) = k_R f(u_R), \tag{4.8}$$

where the subscripts L and R refer to limits from the left and right, respectively, at the jump in k.

LEMMA 4.4. *Let $F(u) = \sigma(u-u^*)(f(u)-f(u^*))$. For a pair of states u_L, u_R satisfying the Rankine Hugoniot condition 4.8,*

$$k_R F(u_R) - k_L F(u_L) \leq |k_R - k_L| f(u^*) \quad \Leftrightarrow \quad f'_-(u_L)f'_+(u_R) = 0. \tag{4.9}$$

Proof. Take $k_L \leq k_R$; the other case is similar. By considering the various cases for a pair of states (u_L, u_R) which satisfy 4.8, the following relationships result:

1. $u_R > u^* > u_L \Rightarrow k_R F(u_R) - k_L F(u_L) = k_R(f_R - f^*) + k_L(f_L - f^*) \leq 0$,
2. $u_R, u_L \geq u^* \Rightarrow k_R F(u_R) - k_L F(u_L) = (k_L - k_R) f^*$,
3. $u_R, u_L \leq u^* \Rightarrow k_R F(u_R) - k_L F(u_L) = (k_R - k_L) f^*$,
4. $u_R < u^* < u_L \Rightarrow k_R F(u_R) - k_L F(u_L) = k_R(f^* - f_R) + k_L(f^* - f_L) > 0$.

The first three cases cover the situation where the equation on the right side of 4.9 is satisfied. In each of those cases, the inequality on the left side of 4.9 also holds. Case 4 is the only case where the right side of 4.9 fails. In that situation, the left side of 4.9 also fails, since $u_L > u^* \Rightarrow f_L < f^*$, and so

$$k_R(f^* - f_R) + k_L(f^* - f_L) = (k_R + k_L) f^* - 2k_L f_L > (k_R - k_L) f^*.$$

□

The entropy inequalities 4.1 and 4.6 yield the following fact concerning the satisfaction of geometric entropy conditions.

THEOREM 4.5. *In addition to the assumptions of Proposition 4.3, assume that the limit solution u is piecewise C^1. Suppose that k jumps from k_L to k_R at $x = \xi_i$, and u is C^1 in some neighborhood of the point (ξ_i, t_0) on each side of $x = \xi_i$. With the notation $u_L = u(\xi_i^-, t_0)$ and $u_R = u(\xi_i^+, t_0)$,*

$$f'_-(u_L) f'_+(u_R) = 0. \tag{4.10}$$

For a discontinuity located at (x_0, t_0) away from the jumps in k, assume that u is C^1 in some neighborhood of (x_0, t_0) on each side of a C^1 curve $x = \gamma(t)$. With $u_L = u(x_0^-, t_0)$ and $u_R = u(x_0^+, t_0)$, the following standard entropy condition holds:

$$k f'(u_L) > s > k f'(u_R), \tag{4.11}$$

where the shock speed $s = \gamma'(t_0)$ satisfies $s = k(f(u_R) - f(u_L))/(u_R - u_L)$.

Proof. For 4.10, let $N(\xi_i, t_0)$ be a neighborhood of the type described in the statement of the theorem. Let $\psi^\epsilon \geq 0$ be a smooth test function with support in the rectangle Ω^ϵ centered at (ξ_i, t_0), with width ϵ, and extending backward and forward in time from t_0 by an amount τ. Assume that ϵ and τ are small enough that $\Omega^\epsilon \subset N(\xi_i, t_0)$. A standard test function calculation, applied to 4.6, gives

$$\int_{t_0-\tau}^{t_0+\tau} \psi^\epsilon(\xi_i, t)([kF] - f^*|k_R - k_L|) dt \leq f^* \int_{\Omega^\epsilon} (|k'| - \sigma(u - u^*) k') \psi^\epsilon dx dt,$$

where $[kF] = k_R F(u_R) - k_L F(u_L)$, $F(u) = \sigma(u - u^*)(f(u) - f^*)$. The integral over Ω^ϵ can be made arbitrarily small by shrinking the width ϵ of the rectangle. It follows that $k_R F(u_R) - k_L F(u_L) \leq f^*|k_R - k_L|$. The entropy condition 4.10 then follows from Lemma 4.4. The geometric entropy condition 4.11 follows directly from Proposition 4.1. For $k \in C^1$, it is well known that if 4.1 holds for all $c \in \mathbf{R}$, then the geometric entropy condition 4.11 is satisfied [8]. □

The geometric entropy condition 4.11 is the usual geometric entropy condition satisfied by shocks in the smooth k case. It requires that characteristics on both sides of the shock extend toward the x-axis when followed backward in time from the shock. The geometric entropy condition 4.10 requires that the characteristics on at least one side extend toward the x-axis.

In the constant-k setting, the author of [16] established L^1-contractiveness of piecewise C^1 solutions which satisfy the geometric entropy condition 4.11. The following theorem is presented as evidence that limits of the scheme 1.4 are the physically

relevant solutions. It extends the theorem in [16] to the case of piecewise constant k, assuming that the additional geometric entropy condition 4.10 is satisfied.

THEOREM 4.6. *In addition to the previous assumptions, let k be piecewise constant, with finitely many jumps, located at $\xi_1 < \xi_2 < \ldots < \xi_M$. Let u and v be piecewise C^1 weak solutions of 1.1 satisfying the geometric entropy conditions 4.10 and 4.11. Assume that the initial data u_0 and v_0 are piecewise C^1, and u_0, $v_0 \in L^1$. Then, for $t \geq 0$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_1 \leq \|u_0 - v_0\|_1. \tag{4.12}$$

Proof. Following [16], the approach is to show that the time derivative of the integral on the left side of 4.12 is nonpositive. That integral is broken up into integrals over segments where $u - v$ does not change sign. This decomposition yields

$$\|u(\cdot,t) - v(\cdot,t)\|_1 = \sum_i \sigma_i \int_{x_i}^{x_{i+1}} (u(x,t) - v(x,t))dx, \tag{4.13}$$

where $\sigma_i = \sigma(u - v)$, which is constant within the interval (x_i, x_{i+1}). Differentiating 4.13 with respect to time results in terms of the form

$$\sigma_i \frac{d}{dt} \int_{x_i}^{x_{i+1}} (u - v)dx = \sigma_i \Big[\dot{x}(u - v) - (kf(u) - kf(v))\Big]_{x_i}^{x_{i+1}}. \tag{4.14}$$

In [16], it is noted that 4.14 holds even if there are shocks in the interior of (x_i, x_{i+1}). For essentially the same reason, i.e., the Rankine-Hugoniot condition 4.8, the relationship 4.14 holds even if one or more of the points ξ_j lie within (x_i, x_{i+1}). For x_i (or x_{i+1}) away from the jumps in k, the contribution to 4.14 from x_i (or x_{i+1}) is nonpositive, since the argument in [16] applies in this case, due to the geometric entropy condition 4.11. The only case remaining is where $u - v$ changes sign at a jump in k, say $x_i = \xi_j$. Let $k_L = k(\xi_j^-)$, $k_R = k(\xi_j^+)$, and assume that $k_L < k_R$, the other case being similar. There are contributions to 4.13 from two consecutive terms of the form 4.14. Taking into account that $\dot{x} = 0$ along the line $x = \xi_j$, the total contribution is

$$\sigma_i \Big[-k_L(f(u(x)) - f(v(x)))\Big]_{\xi_j^-} + \sigma_{i+1} \Big[k_R(f(u(x)) - f(v(x)))\Big]_{\xi_j^+}. \tag{4.15}$$

The situation where $\sigma_i = 0$ or $\sigma_{i+1} = 0$ can be eliminated, since then the Rankine-Hugoniot condition 4.8 implies that both σ_i and σ_{i+1} vanish. Take the case where $u > v$ in (x_{i-1}, x_i), $u < v$ in (x_i, x_{i+1}), i.e., $\sigma_i = 1$, $\sigma_{i+1} = -1$. The other case is similar. Then 4.15 becomes

$$k_L f(v_L) + k_R f(v_R) - k_L f(u_L) - k_R f(u_R). \tag{4.16}$$

In order for there to be a sign change, at least one of $u_R < u_L$, $v_R > v_L$ must hold. Take the case where $u_R < u_L$. With $k_L < k_R$ and $u_R < u_L$, the geometric entropy condition 4.10 and Rankine-Hugoniot condition 4.8 require that $u_L < u^*$. This is easily verified by viewing $(u_L, k_L f(u_L))$ and $(u_R, k_R f(u_R))$ as being determined by the intersection of a horizontal line with the graphs of $k_L f(u)$ and $k_R f(u)$. Since $v_L < u_L \leq u^*$, and f is increasing in $(0, u^*)$, it is clear that $f(v_L) < f(u_L)$. Then using 4.8, the expression 4.16 is equal to

$$2k_L(f(v_L) - f(u_L)) < 0.$$

Next, take the case where $v_R > v_L$. With $k_L < k_R$ and $v_R > v_L$, the Rankine-Hugoniot condition 4.8 requires that $v_R > u^*$. If $v_L \geq u^*$, then $u_L > v_L \Rightarrow u_L > u^*$. The entropy condition 4.10 then requires that $u_R \geq u^*$. Then, with 4.8, $u_L > v_L \Rightarrow u_R > v_R$, which is a contradiction, so it must be that $v_L < u^*$. If also $u_L < u^*$, then $v_L < u_L \Rightarrow f(v_L) < f(u_L)$, in which case 4.16 reduces to

$$2k_L(f(v_L) - f(u_L)) < 0.$$

If $u_L > u^*$, the entropy condition 4.10 requires that $u_R \geq u^*$. Then, $u_R < v_R \Rightarrow f(u_R) > f(v_R)$. In this case 4.16 equals

$$2k_R(f(v_R) - f(u_R)) < 0.$$

□

REFERENCES

[1] M. G. Crandall, A. Majda, *Monotone Difference Approximations for Scalar Conservation Laws*, Math. Comp., 34 (1980), pp. 1–21.

[2] B. Engquist, S. Osher, *One sided difference approximations for nonlinear conservation laws*, Math. Comp., 36 (1980), pp. 45–75.

[3] S.K. Godunov, *A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics* , Math. USSR Sbornik, 47 (1959), pp. 271–290.

[4] A. Harten, *On a class of high resolution total-variation-stable finite difference schemes*, SIAM J. Numer. Anal., 21 (1984), pp. 1–23.

[5] A. Harten, J.M. Hyman, P.D. Lax, *On finite difference approximations and entropy conditions for shocks*, Comm. Pure Appl. Math., 29 (1976), pp. 297–322.

[6] R.A. Klausen, N.H. Risebro, *Stability of conservation laws with discontinuous coefficients*, J. Differential Equations, 157 (1999), pp. 41–60.

[7] C. Klingenberg, N.H. Risebro, *Convex conservation laws with discontinuous coefficients. Existence, uniqueness, and asymptotic behavior*, Comm. Partial Differential Equations, 20, (1995), pp. 1959–190.

[8] S.N. Kruzkov, *First order quasilinear equations in several independent variables*, Math. USSR Sbornik, 10, (1970), pp. 217–243.

[9] N. N. Kuznetsov, *Accuracy of some approximate methods for computing the weak solutions of a first-order quasilinear equation*, USSR Comp. Math. Phys., 16, 6 (1976), pp. 105–119.

[10] P.D. Lax, B Wendroff, *Systems of conservation laws*, Comm. Pure Appl. Math., 23 (1960), pp. 217-237.

[11] R. J. Leveque, *Numerical methods for conservation laws*, Birkhauser Verlag, Boston, MA, 1992.

[12] L. Lin, J.B. Temple, J. Wang, *A comparison of convergence rates for Godunov's method and Glimm's method in resonant nonlinear systems of conservation laws*, SIAM J. Numer. Anal., 32 (1995), pp. 824-840.

[13] L. Longwei, J.B. Temple, W. Jinghua, *Suppression of oscillations in Godunov's method for a resonant non-strictly hyperbolic system*, SIAM J. Numer. Anal., 32 (1995), pp. 841-864.

[14] S. J. Osher, *Riemann solvers, the entropy condition, and difference approximations*, SIAM J. Numer. Anal., 21 (1984), pp. 217–235.

[15] S. J. Osher, S. Chakravarthy, *High resolution schemes and the entropy condition*, SIAM J. Numer. Anal., 21 (1984), pp. 955–984.

[16] B. K. Quinn, *Solutions with shocks: An example of an L_1-contractive semigroup*, Comm. Pure Appl. Math., 24 (1971), pp. 125-132.

[17] R. Sanders, *On convergence of monotone difference schemes with variable spatial differencing*, Math. Comp., 40 (1983), pp. 91–106.

[18] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, NY, 1983.

[19] B. Temple, *Global solution of the cauchy problem for a class of 2x2 nonstrictly hyperbolic conservation laws*, Adv. in Appl. Math., 3 (1982), pp. 335–375.

[20] A. Tveito and R. Winther, *The solution of nonstrictly hyperbolic conservation laws may be hard to compute*, SIAM J. Sci. Comput., 16 (1995), pp. 320–329.