A difference scheme for an inhomogeneous traffic flow model

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Initial value problem

$$\begin{cases} \phi_t + f(a(x),\phi)_x = 0, \quad t > 0,\, x \in \mathbb{R} \\ \phi(x,0) = \phi_0(x) \end{cases}$$

f = flux, ϕ = car density, v=velocity, a = jam density

The flux

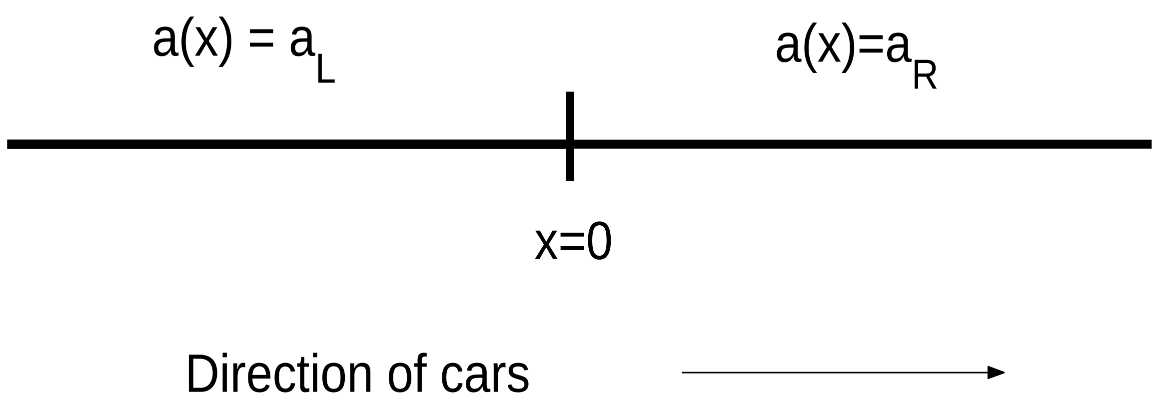
$$\begin{cases} f(a,\phi) = \phi v(\phi/a), \quad a > 0, \quad \phi \in [0,a] \\ v : [0,1] \mapsto [0, v_{\max}], \quad v(1) = 0, \text{ decreasing} \end{cases}$$

Example flux

$$v(\phi) = v_{\max} \cdot (1-\phi)$$

$$f(a,\phi) = \phi\, v_{\max} \cdot (1-\phi/a)$$

a(x) constant except for jump at x=0



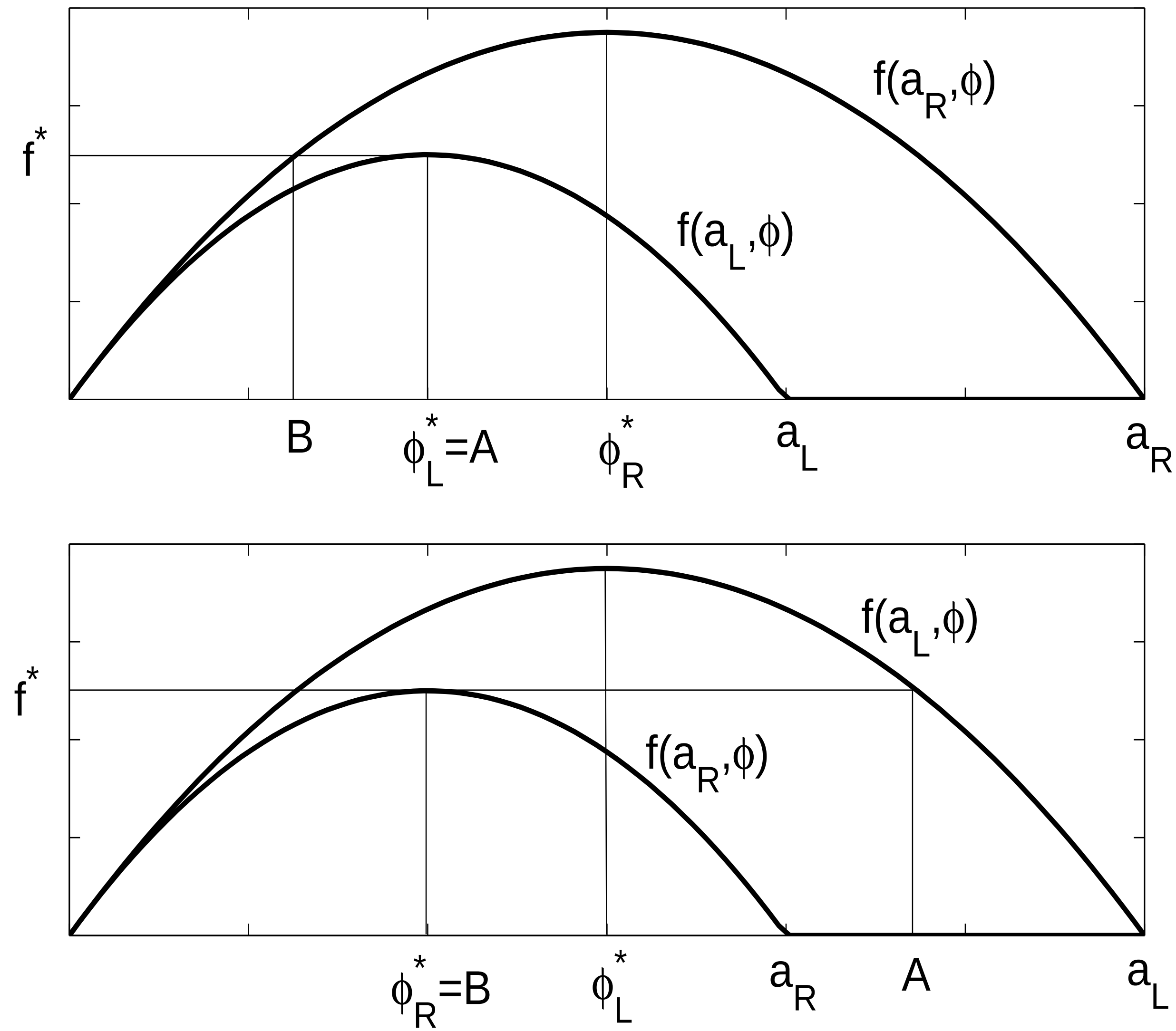
Jump models change in road, eg. change in number of lanes

Conservation law with discontinuous flux

Scalar conservation law with discontinuous flux (partial list)

- Gimse, Risebro 1992
- Klingenberg, Risebro 1995
- Lin, Temple, Wang 1995
- Diehl 1996
- Ostrov 2002
- **Jin, Zhang 2003 - Studied this specific traffic flow model**
- Seguin, Vovelle 2003
- Adimuthi, Jaffre, Veerappa Gowda 2004
- Adimurthi, Mishra, Veerappa Gowda 2004
- Audusse, Perthame 2005
- Bachmann, Vovelle 2006

The flux $f(a, \phi)$



Special steady solution:

$$\phi(x) = \begin{cases} A, & x < 0 \\ B, & x > 0 \end{cases}$$

(A, B) = ”connection” (Adimurthi, Mishra, Veerappa Gowda)

Other connections exist - different entropy solutions

Chosen connection can be justified based on driver behavior

Uniqueness

Classical Kruzkov theory not applicable

We define Entropy Solution using (partially) **Adapted Entropy Inequality**

$$\int_{t>0}\int_{x\in\mathbb{R}}\Big\{|\phi-c|\,\psi_t+F(a,\phi,c)\psi_x\Big\}dx\,dt\geq 0$$

for all smooth test functions $\psi \geq 0$

$$c(x)=\begin{cases} A, & x<0 \\ B, & x>0 \end{cases}$$

$$F(a,\phi,c)=\mathrm{sign}(\phi-c)\left(f(a,\phi)-f(a,c)\right)$$

Adapted Entropy Concept

Baiti, Jenssen 1997

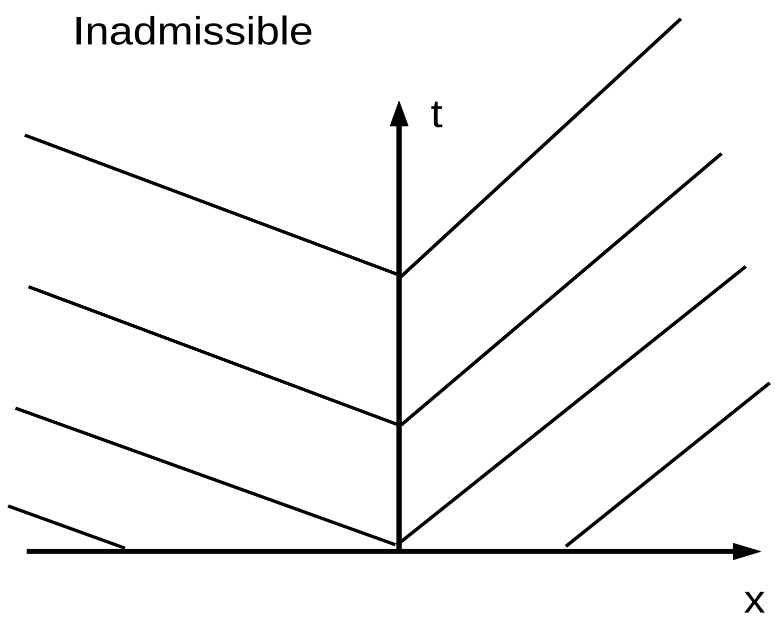
Audusse, Perthame 2005

Chen, Even, Klingenberg 2006

Jump conditions at $x = 0$

$$f(a_R, \phi_R) = f(a_L, \phi_L) \quad \text{(Rankine-Hugoniot)}$$

$\phi_L > A$, $\phi_R < B$ is inadmissible (Entropy Condition)

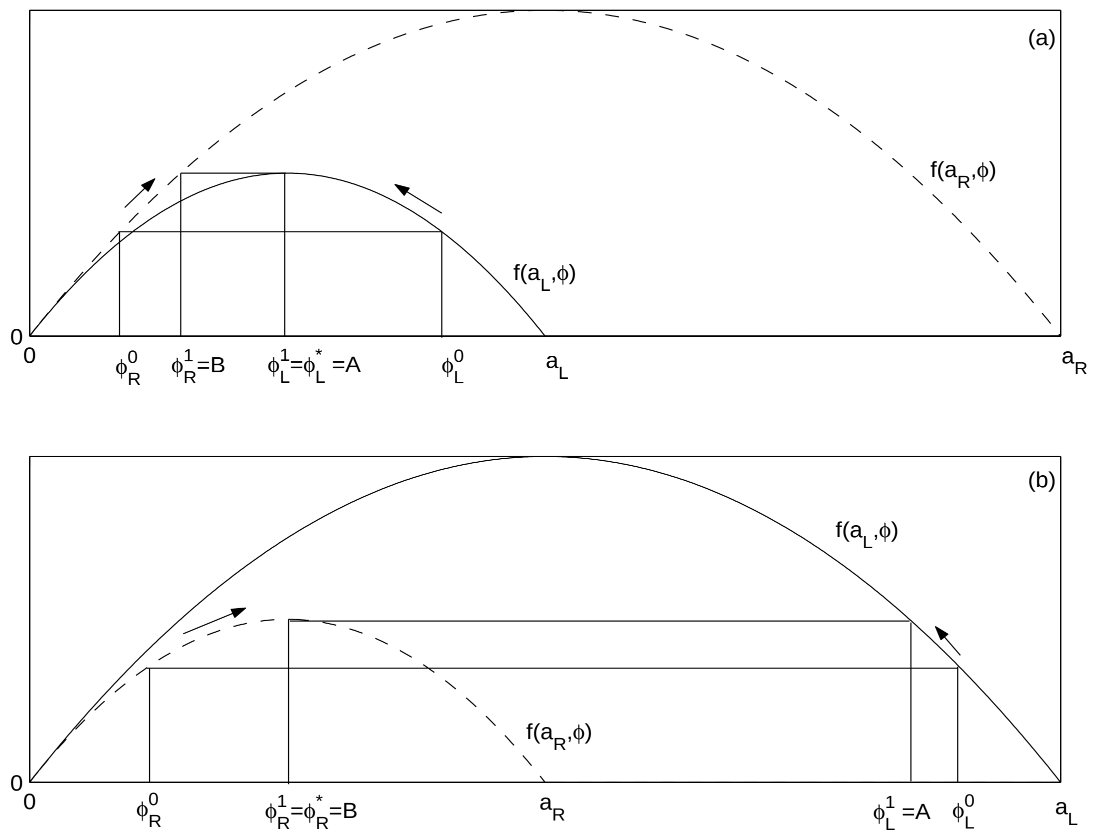


Characteristics must lead back to x-axis on at least one side of jump

Jump conditions and driver behavior

"Driver's ride impulse" (Ansorge - 1990) not applicable

Our assumption: Drivers adjust speed based on relative speed of traffic directly ahead of them.



Uniqueness theorem for entropy solutions

If ϕ and $\hat{\phi}$ are a pair of entropy solutions with initial data ϕ_0 and $\hat{\phi}_0$, then

$$\int_{\mathbb{R}} \left|\phi(x,t) - \hat{\phi}(x,t)\right| \, dx \leq \int_{\mathbb{R}} \left|\phi_0(x) - \hat{\phi}_0(x)\right| \, dx$$

Proof is by adaptation of Kružkov ”doubling of variables”

Finite Difference Schemes

$$\Phi_j^{n+1} = \Phi_j^n - \lambda \left(h_{j+\frac{1}{2}}^n - h_{j-\frac{1}{2}}^n \right), \quad \lambda = \Delta t / \Delta x$$

$$\Phi_j^n \approx \phi(x_j, t^n), \quad x_j = j\Delta x, \quad t^n = n\Delta t$$

Numerical Flux

$$h_{j+\frac{1}{2}}^n = \begin{cases} \bar{f}\left(a_L, \Phi_{j+1}^n, \Phi_j^n\right), & j < 0 \\ \bar{f}_{\text{int}}\left(a_R, a_L, \Phi_{j+1}^n, \Phi_j^n\right), & j = 0 \\ \bar{f}\left(a_R, \Phi_{j+1}^n, \Phi_j^n\right), & j > 0 \end{cases}$$

Away from interface, we use standard 2-point monotone flux:

$$\begin{cases} \bar{f}\left(a_L, \Phi^n_{j+1}, \Phi^n_j\right) \\ \bar{f}\left(a_R, \Phi^n_{j+1}, \Phi^n_j\right) \end{cases}$$

Interface flux:

$$\bar{f}_{\text{int}}\left(a_R, a_L, \Phi^n_{j+1}, \Phi^n_j\right)$$

- Numerical fluxes are monotone: Both $\bar{f}$ and $\bar{f}_{\text{int}}$ are
  - nonincreasing as function of Φ^n_{j+1}
  - nondecreasing as function of Φ^n_j
- $\bar{f}_{\text{int}}(a_R, a_L, B, A) = f(a_R, B) = f(a_L, A)$

Scheme preserves discrete version of $c(x) = \begin{cases} A, & x < 0 \\ B, & x > 0 \end{cases}$

Three versions of first order scheme:

- Hilliges-Weidlich
- Godunov
- Engquist-Osher

Properties:

- Monotone
- Conservative (cars are conserved)
- Preserve important (discrete) steady solutions
- Satisfy discrete entropy inequalities
  - Standard ones away from interface
  - Discrete adapted entropy inequality at interface

- Enforce bounds:

$$\Phi_j^n \in \begin{cases} [0, a_L], & j \leq 0, \\ [0, a_R], & j > 0. \end{cases}$$

- Local bound on spatial variation

Theorem: With all three versions of first order scheme, have convergence to unique entropy solution.

First order schemes easily extended to (formal) second order accuracy:

- MUSCL spatial differencing
- Runge-Kutta time differencing

Hilliges-Weidlich flux

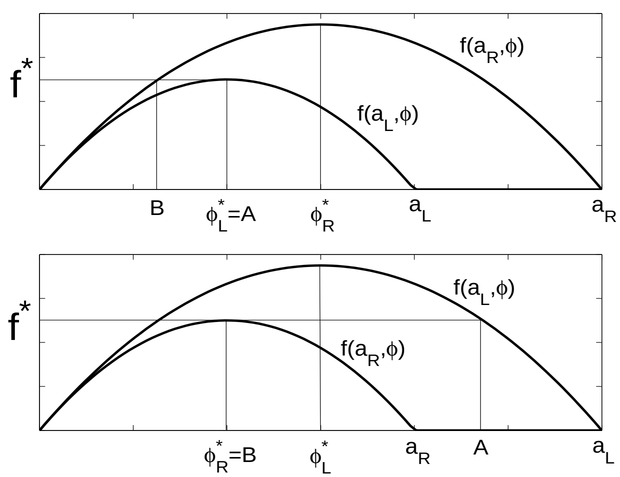
Away from interface ($j \neq 0$):

$$\bar{f}_{j+\frac{1}{2}} = \Phi_j^n v(\Phi_{j+1}^n / a_{j+1})$$

This was used by Hilliges and Weidlich for discrete model of traffic flow, but not considered as a numerical flux for conservation laws

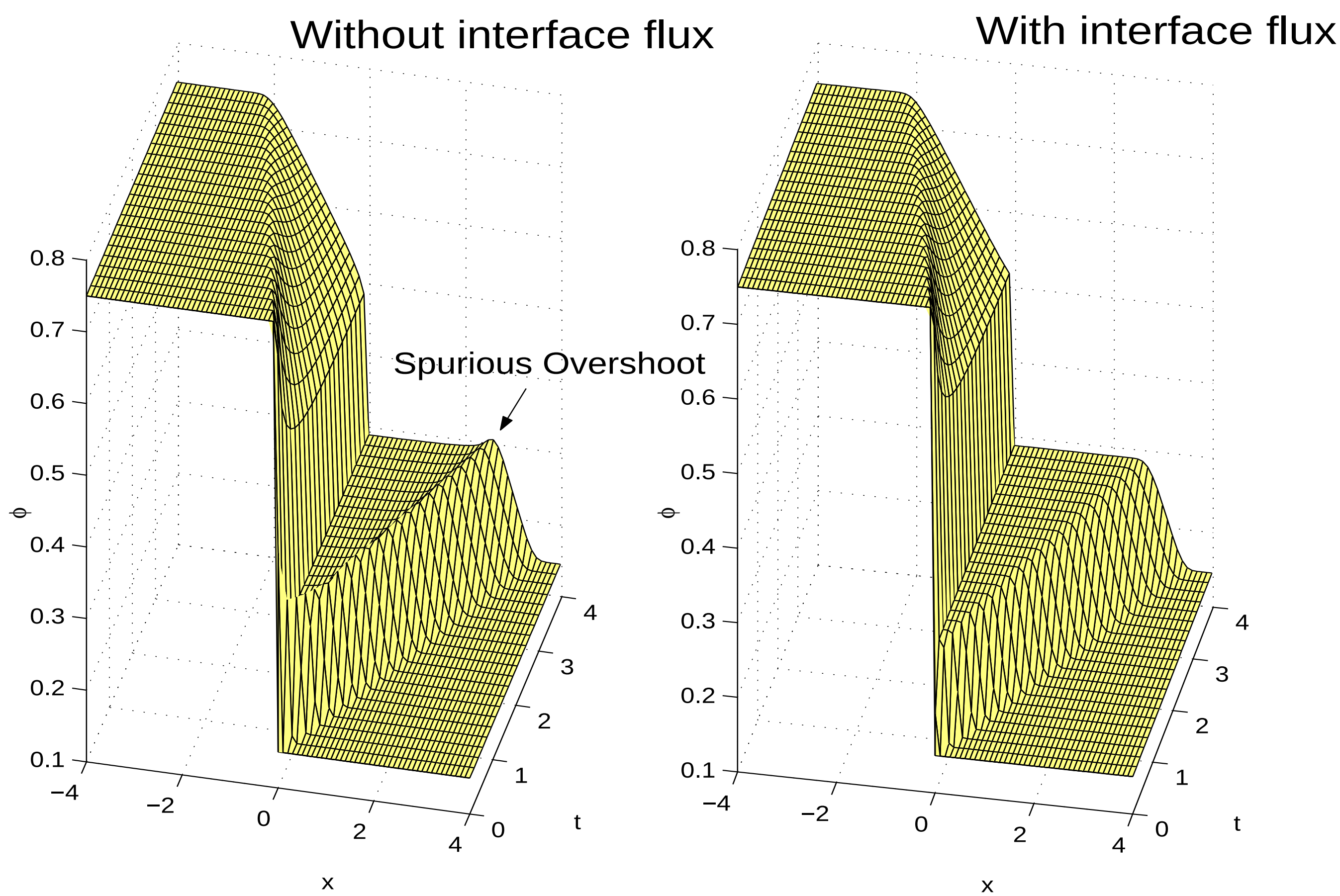
Interface flux ($j = 0$):

$$\bar{f}_{\text{int}} = \min\left(\Phi_j^n v(\Phi_{j+1}^n / a_{j+1}), f^*\right)$$



Hilliges-Weidlich flux - Riemann Problem Example

$$\bar{f}_{j+\frac{1}{2}} = \Phi_j^n v(\Phi_{j+1}^n / a_{j+1}), \quad \bar{f}_{\text{int}} = \min\left(\Phi_j^n v(\Phi_{j+1}^n / a_{j+1}), f^*\right)$$



Godunov flux

Away from interface ($j \neq 0$):

$$\bar{f}_{j+\frac{1}{2}} = \begin{cases} \min\left\{f(a,\phi) : \Phi_j^n \leq \phi \leq \Phi_{j+1}^n\right\}, & \Phi_j^n \leq \Phi_{j+1}^n \\ \max\left\{f(a,\phi) : \Phi_{j+1}^n \leq \phi \leq \Phi_j^n\right\}, & \Phi_j^n \geq \Phi_{j+1}^n \end{cases}$$

Interface flux ($j = 0$):

$$\bar{f}_{\text{int}} = \min\left[f(a_L, \min(\Phi_j^n, \phi_L^*)), f(a_R, \max(\Phi_{j+1}^n, \phi_R^*))\right]$$

Godunov interface flux discovered several times

- Lebacque
- Daganzo
- Adimurthi, Jaffre, Veerappa Gowda
- Jin, Zhang

Engquist-Osher flux

Away from interface ($j \neq 0$):

$$\bar{f}_{j+\frac{1}{2}} = \frac{1}{2}\left(f(a,\Phi_{j+1}^n) + f(a,\Phi_j^n)\right) - \frac{1}{2}\int_{\Phi_j^n}^{\Phi_{j+1}^n} \left|\partial_\phi f(a,\phi)\right| \, d\phi$$

Interface flux ($j = 0$):

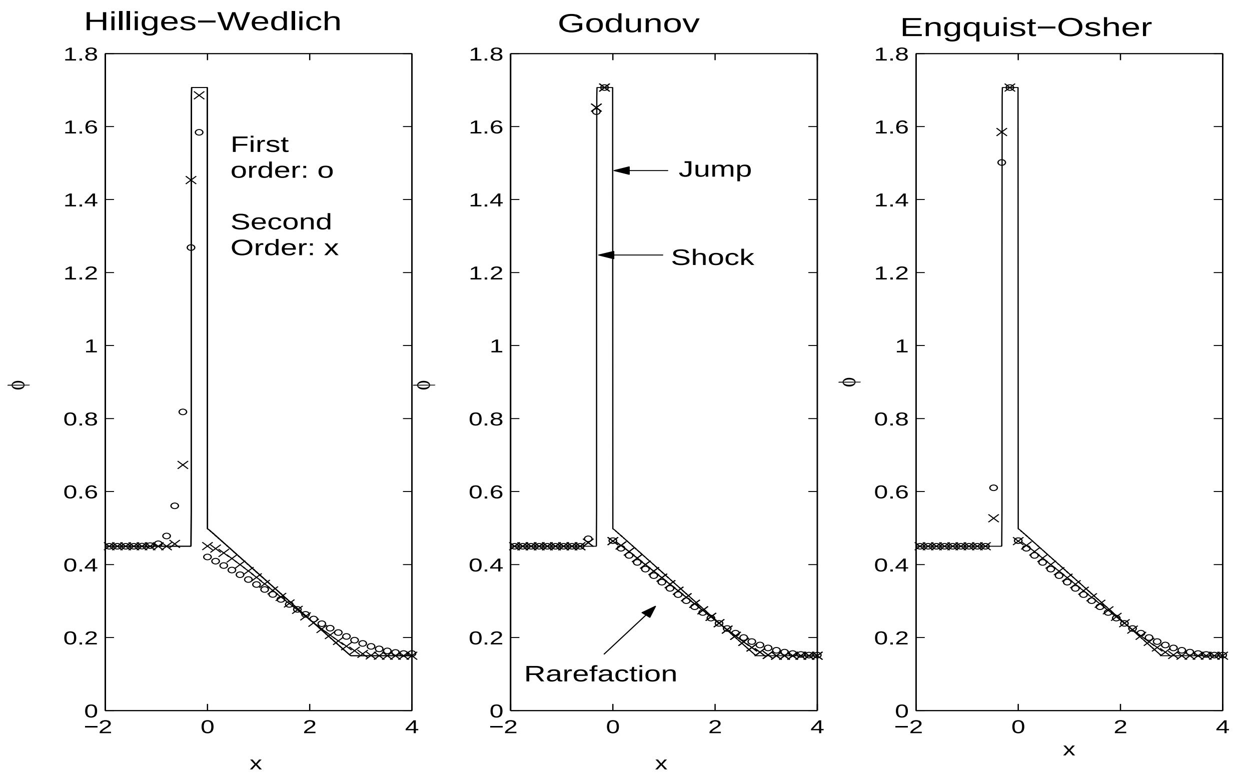
$$\bar{f}_{\text{int}} = \frac{1}{2}\left(\tilde{f}(a_R,\Phi_{j+1}^n) + \tilde{f}(a_L,\Phi_j^n)\right) - \frac{1}{2}\left(\int_B^{\Phi_{j+1}^n} \left|\partial_\phi \tilde{f}(a_R,\phi)\right| \, d\phi - \int_A^{\Phi_j^n} \left|\partial_\phi \tilde{f}(a_L,\phi)\right| \, d\phi\right)$$

$$\tilde{f}(a_R,\phi) = \min\left(f(a_R,\phi), f^*\right)$$

$$\tilde{f}(a_L,\phi) = \min\left(f(a_L,\phi), f^*\right)$$

A Riemann Problem with $f(a,\phi) = \phi(1-\phi/a)$

$$\phi_0(x) = \begin{cases} 0.45 & \text{for } x < 0, \\ 0.15 & \text{for } x > 0, \end{cases} \qquad a(x) = \begin{cases} 2 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases} \tag{1}$$



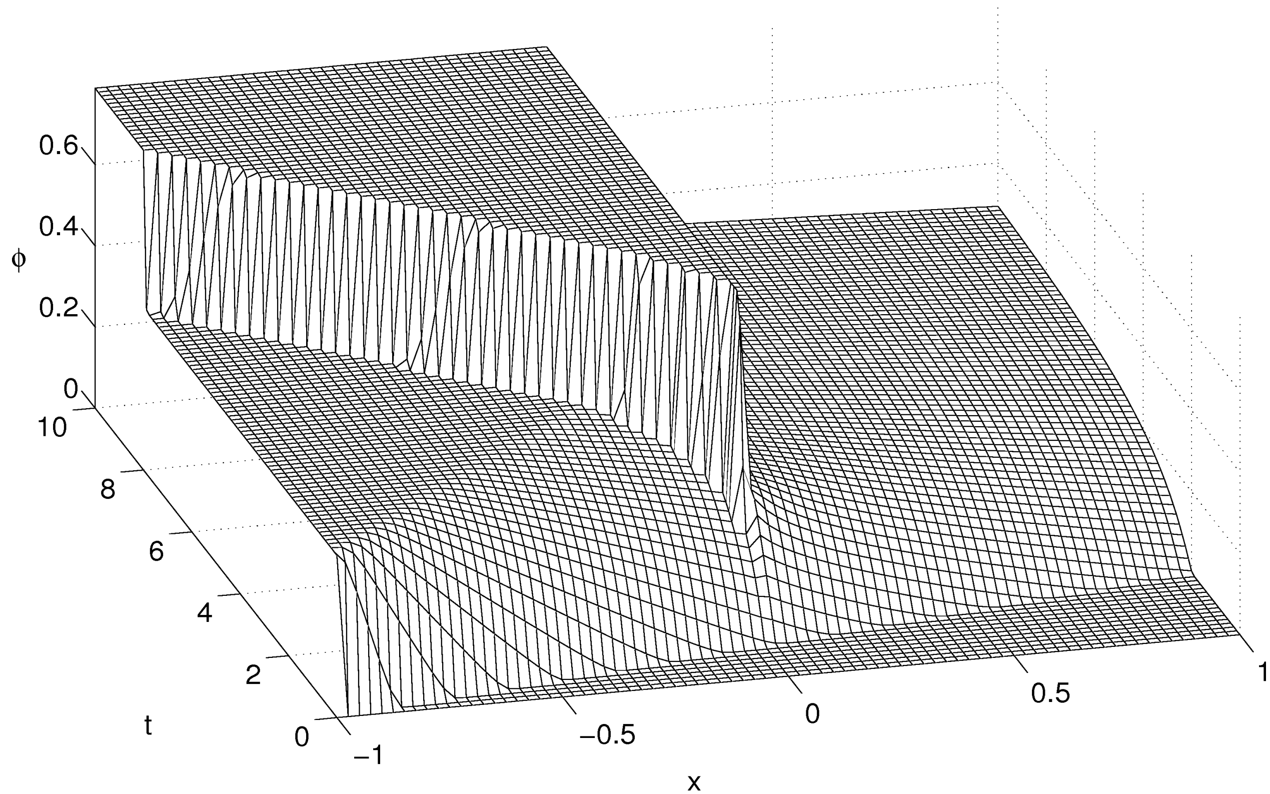
L^1 errors for a Riemann problem (at a fixed time):

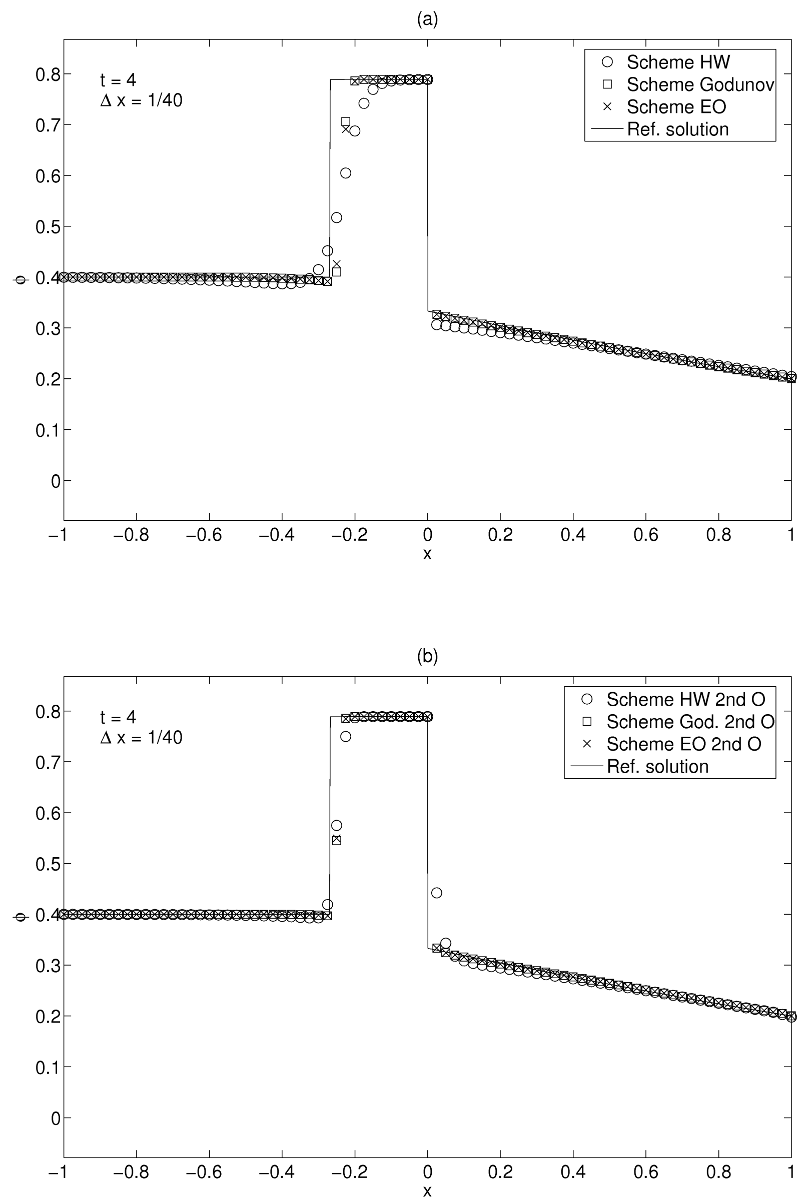
Table 1: Example 2.

	HW	Godunov	EO
Order	Error	Error	Error
1st Order	$13.5\text{e}{-3}$	$8.90\text{e}{-3}$	$8.91\text{e}{-3}$
2nd Order	$8.03\text{e}{-3}$	$7.98\text{e}{-3}$	$8.00\text{e}{-3}$

Bottleneck problem (from book by Garavello and Piccoli - 2006)

$$\phi_0(x) = \begin{cases} 0.4 & \text{for } x < -1, \\ 0 & \text{for } x > -1, \end{cases} \qquad a(x) = \begin{cases} 1 & \text{for } x < 0, \\ 2/3 & \text{for } x > 0. \end{cases} \tag{2}$$




(a)
t = 4
Δ x = 1/40
Scheme HW
Scheme Godunov
Scheme EO
Ref. solution
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
–1
–0.8
–0.6
–0.4
–0.2
0
0.2
0.4
0.6
0.8
1
x
(b)
t = 4
Δ x = 1/40
Scheme HW 2nd O
Scheme God. 2nd O
Scheme EO 2nd O
Ref. solution
x