Entropy condition for multiclass traffic flow with discontinuous flux

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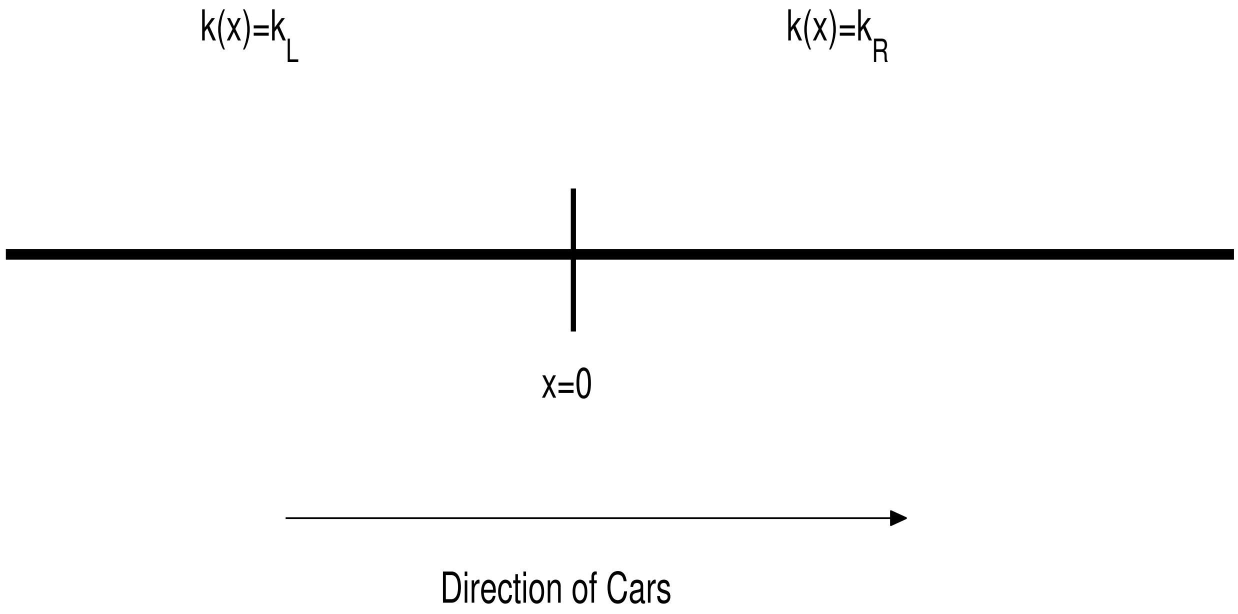
Initial value problem - system of conservation laws with spatially discontinuous flux

$$\begin{aligned}
&\vec{\phi}_t + \Big(k(x)\vec{f}(\vec{\phi})\Big)_x = \vec{0} \\
&\vec{\phi} = (\phi_1, \dots, \phi_m), \quad \vec{f} = (f_1(\vec{\phi}), \dots, f_m(\vec{\phi})) \\
&f_i(\phi) = \phi_i v_i
\end{aligned} \tag{1}$$

- To be solved in $\Pi := \{(x,t) | x \in \mathbb{R},\, t \geq 0\}$
- Initial data $\vec{\phi}(x,0) = \vec{p}(x)$
- Models traffic flow in one direction on a highway
- Mutliple classes of drivers/vehicles



- Jump in $k(x)$ models change in road conditions

$$\underline{\partial_t \phi_i + \partial_x (k(x)\phi_i v_i) = 0, \quad i = 1, \dots, m} \tag{2}$$

- ϕ_i is density of ith class of vehicles/drivers
- $\phi := \sum_{i=1}^{m} \phi_i$ is total density; Assume $0 \leq \phi \leq 1$
- $k(x)v_i$ is velocity of ith class
- $v_i = V_i \psi(\phi)$, V_i = positive constant

$$\underline{\partial_t \phi_i + \partial_x (k(x)\phi_i v_i) = 0, \quad i = 1, \ldots, m} \tag{3}$$

- Velocity of class i: $v_i = V_i \psi(\phi)$, V_i = positive constant
- $\psi(\phi)$ decreasing. $\phi(1) = 0$
- $k(x)$ is piecewise constant:

$$k(x) = \begin{cases} k_L, & x < 0 \\ k_R, & x > 0 \end{cases} \tag{4}$$

- The coefficient k scales the velocity of all classes by the same amount
- Problem is harder if k varies with driver/vehicle class

• Solutions generally discontinuous

• Even with constant k and smooth initial data

• Forces us to settle for a **weak solution**:

i.e., a bounded measurable function $\vec{\phi}$ satsfying

$$\int_{t>0}\int_{\mathbb{R}} \left(\eta_t\vec{\phi} + \eta_x k\vec{f}(\vec{\phi})\right) dx\, dt + \int_{\mathbb{R}} \eta(x,0)\vec{\phi}(x,0)dx = 0 \qquad (5)$$

for any smooth test function $\eta(x,t)$ with compact support contained in $\Pi := \{(x,t)|x \in \mathbb{R},\, t \geq 0\}$

Difference scheme (semi-discrete version)

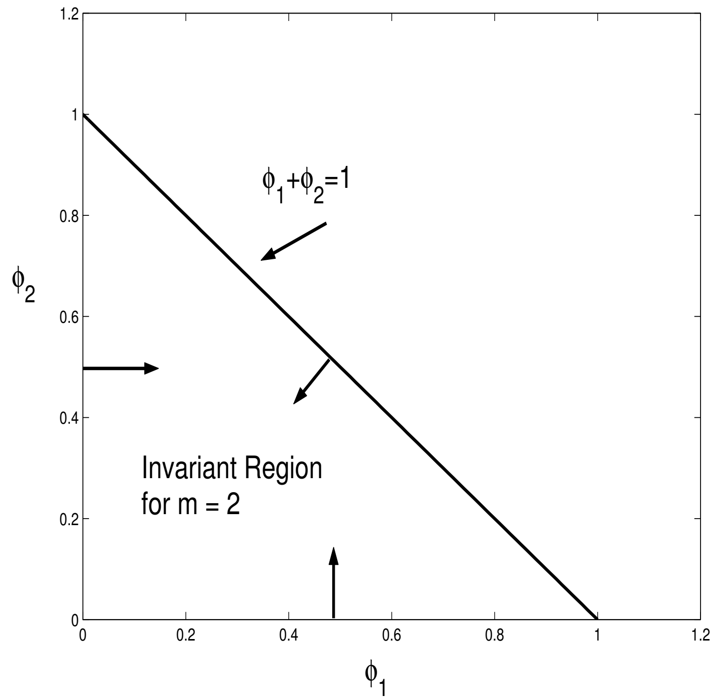
$$\frac{d}{dt}\phi_{i,j} = -\frac{1}{\Delta x}\Delta_-^x h_{i,j+\frac{1}{2}}, \quad i = 1, \dots, m \tag{6}$$

- Numerical flux:

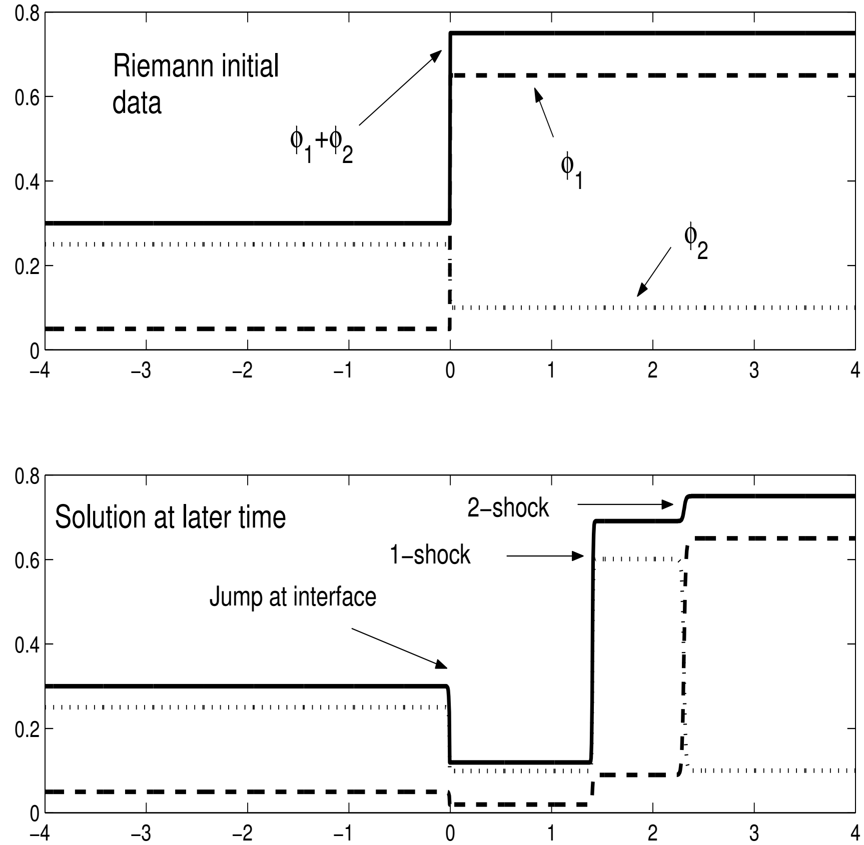
$$\begin{aligned} h_{i,j+\frac{1}{2}} &= \phi_{i,j} V_i k_{j+1} \psi(\phi_{j+1}) \\ &= \text{density}_j \times \text{velocity}_{j+1} \end{aligned} \tag{7}$$

- Studied for fully discrete (no PDE) scalar traffic flow by Hilliges, Wiedlich.
- Proposed more recently for systems of conservation laws by Bürger, García, Karlsen, T (BGKT2006)
- Convergence in scalar case analyzed in detail

- Scheme does not require characteristic decomposition (eigenvalues, eigenvectors)
- Characteristic decomposition for this system not straightforward
- Scheme preserves natural invariant region (BGKT 2006)
- Easy to make second order version (MUSCL and Runge-Kutta)



What the solution of a Riemann problem looks like:



Plots generated with first order accurate version of scheme.

$$\psi(\phi) = 1 - \phi, \quad V_1 = 1, \quad V_2 = 3/4, \quad k_L = 1/2, \quad k_R = 1 \tag{8}$$

Scalar conservation law with discontinuous flux: partial list

- Gimse, Risebro (1992)
- Klingenberg, Risebro
- Lin, Temple, Wang
- Diehl
- Ostrov
- Bürger, García, Karlsen, T
- Jin, Zhang
- Seguin, Vovelle
- Adimurthi, Jaffre, Mishra, Veerappa Gowda
- Audusse, Perthame
- Bachmann, Vovelle

System of conservation laws for multiclass traffic: partial list

- Wong, Wong (2002)
- Benzoni-Gavage, Colombo (2003)
- Zhang, Shu, Wong, Wong (2003)
- Zhang, Liu (2003)
- Zhang, Liu, Wong, Dai (2006)
- Donat, Mulet (2007)

Zhang et. al. discuss spatially discontinuous flux for multiclass traffic system

What about entropy satisfaction for limit solutions of scheme?

Two different questions:

1. Entropy satisfaction away from interface

Do limit solutions (*) satisfy the classical Lax shock conditions?

2. Entropy satisfaction for discontinuity at interface (where k jumps)

What conditions are satisfied by limit solutions at the interface?

There is no classical theory for this situation.

(*) We are assuming limit solutions exist.

Away from interface.

Away from interface. ($k =$ constant)

- Smooth solutions of the system satisfy additional conservation law

$$\mathcal{E}(\phi)_t + (k\mathcal{F}(\phi))_x = 0$$

- Convex entropy $\mathcal{E}$ and associated entropy flux $\mathcal{F}$:

$$\mathcal{E}(\vec{\phi}) = \sum_{i=1}^{m} \frac{1}{V_i}\phi_i(\log\phi_i - 1), \quad \mathcal{F}(\vec{\phi}) = \psi(\phi)\sum_{i=1}^{m}\phi_i\log\phi_i - \Psi(\phi)$$

$\Psi'(\phi) = \psi(\phi)$

- Benzoni-Gavage and Colombo (2003)

Away from interface. (k = constant)

- Assume discontinuity moving with speed $dx/dt = s$.
- Rankine-Hugoniot is

$$k\vec{f}(\vec{\phi}_+) - k\vec{f}(\vec{\phi}_-) = s\left(\vec{\phi}_+ - \vec{\phi}_-\right) \tag{9}$$

Not sufficient to single out unique solution

- Lax shock condition: For some index μ, $1 \leq \mu \leq m$

$$\lambda_{\mu-1}(\vec{\phi}_-) < s < \lambda_\mu(\vec{\phi}_-), \qquad \lambda_\mu(\vec{\phi}_+) < s < \lambda_{\mu+1}(\vec{\phi}_+) \tag{10}$$

This is imposed to rule out unwanted solutions.

Away from interface. ($k = \text{constant}$)

• Lax shock condition equivalent to integral **entropy inequality**:

For all smooth test functions $0 \le \rho \in C_0^\infty$

$$\int_{t>0} \int_{\mathbb{R}} \left(\rho_t \mathcal{E}(\vec{\phi}) + \rho_x k \mathcal{F}(\vec{\phi}) \right) dx\, dt \ge 0 \tag{11}$$

• Equivalent **entropy jump condition**:

$$k\mathcal{F}(\vec{\phi}_+) - k\mathcal{F}(\vec{\phi}_-) \le s \left(\mathcal{E}(\vec{\phi}_+) - \mathcal{E}(\vec{\phi}_-) \right) \tag{12}$$

• For this system, entropy jump condition is equivalent to

$$\phi_- \le \phi_+ \tag{13}$$

Away from interface. ($k = $ constant)

- **Lemma:** Diff. scheme satisfies discrete version of entropy inequality:

$$\frac{d}{dt}\mathcal{E}(\vec{\phi}_j) + D_+ \tilde{\mathcal{F}}_{j-\frac{1}{2}} \le 0 \text{ for } x_j \text{ away from interface} \tag{14}$$

- Numerical entropy flux (Osher - early 80's):

$$\tilde{\mathcal{F}}_{j-\frac{1}{2}} := \mathcal{E}_{\vec{\phi}}(\vec{\phi}_j) \cdot \left(\vec{h}_{j-\frac{1}{2}} - k\vec{f}(\vec{\phi}_j)\right) + k\mathcal{F}(\vec{\phi}_j)$$

- **Theorem:** For piecewise continuous limit solutions, integral entropy inequality is satisfied, and discontinuities (away from interface) are classical Lax shocks. They satisfy $\phi_- \le \phi_+$.

- **Brief version:** Scheme captures Lax shocks away from the interface.

At interface.

At interface. (Where k is discontinuous.)

• Rankine-Hugoniot Condition (flux components continuous across interface):

$$k_L V_i \phi_{i,-} \psi(\phi_-) = k_R V_i \phi_{i,+} \psi(\phi_+), \quad i = 1, \dots, m \tag{15}$$

• These imply an additional "scalar Rankine-Hugoniot" condition:

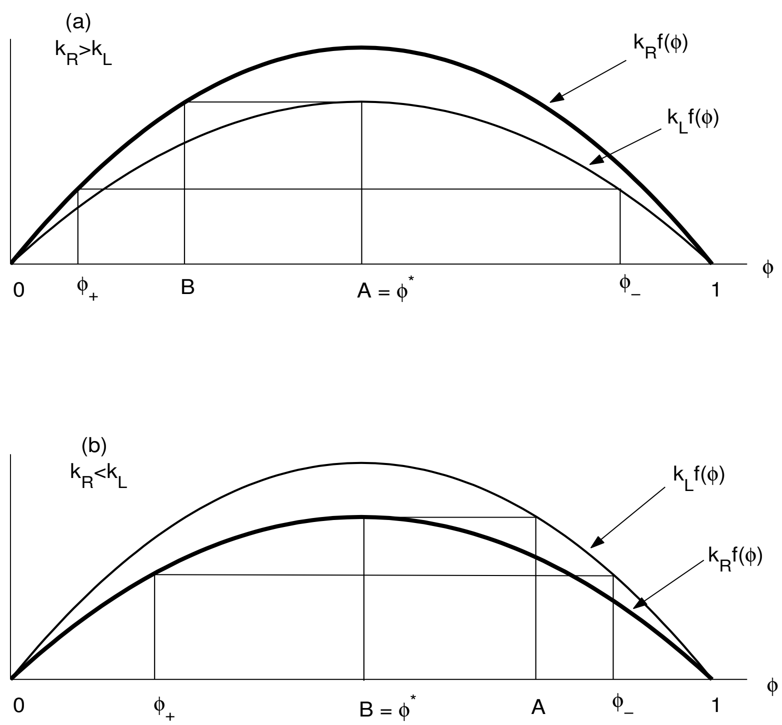
$$k_L \phi_- \psi(\phi_-) = k_R \phi_+ \psi(\phi_+) \tag{16}$$

• This is the Rankine-Hugoniot condition (at interface) for the scalar conservation law

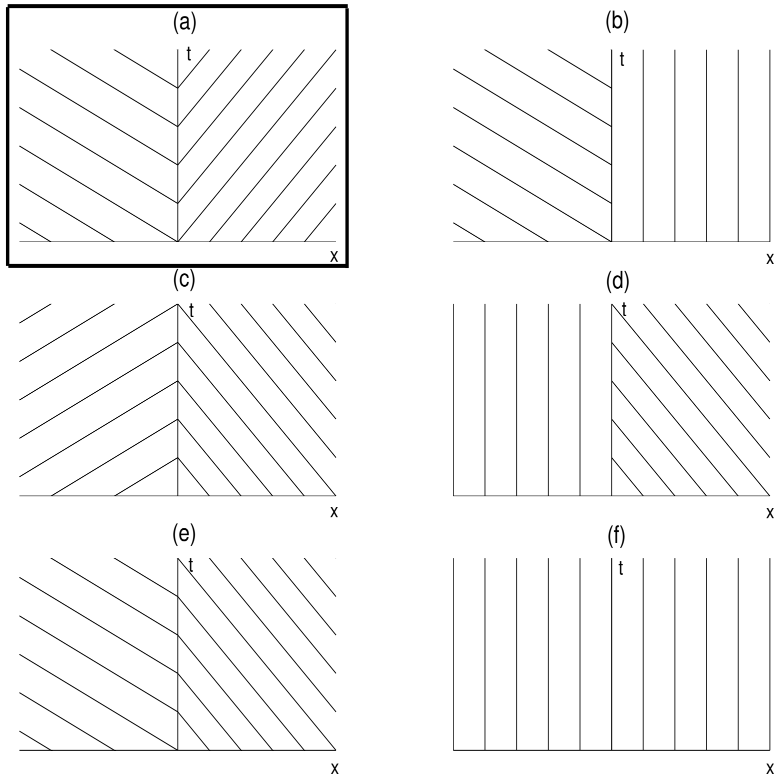
$$\phi_t + (k(x) f(\phi))_x = 0, \quad f(\phi) := \phi \psi(\phi) \tag{17}$$

At interface.

- Entropy jump condition stated in terms of total density ϕ
- Inadmissible jumps (ϕ_-, ϕ_+) are the same as for the scalar conservation law $\phi_t + (k(x)f(\phi))_x = 0$



At interface.



Characterisitics $dx/dt = kf'(u)$ on either side of interface

- Only configuration (a) is inadmissible

At interface.

• We can use the same (!) form of the integral entropy inequality as the "away from interface" situation:

For all smooth test functions $0 \leq \rho \in C_0^\infty$

$$\int_{t>0} \int_{\mathbb{R}} \left(\rho_t \mathcal{E}(\vec{\phi}) + \rho_x k \mathcal{F}(\vec{\phi}) \right) dx\, dt \geq 0 \tag{18}$$

• The following two choices of Ψ give us the desired jump conditions:

$$\Psi(\phi) = \begin{cases} \int_1^\phi \psi(z)\, dz, & k_L < k_R \\ \int_0^\phi \psi(z)\, dz, & k_L > k_R \end{cases} \tag{19}$$

At interface.

- Integral entropy inequality gives entropy jump condition at interface

$$k_R \mathcal{F}(\vec{\phi}_+) - k_L \mathcal{F}(\vec{\phi}_-) \leq 0 \tag{20}$$

- This jump condition implies **interface** entropy jump condition:

$$k_R F(\phi_+) - k_L F(\phi_-) \leq 0 \tag{21}$$

Here F is the scalar entropy flux associated with the scalar entropy E:

$$E(\phi) = \psi(\phi)\left(\log\phi - 1\right), \quad F(\phi) = \psi(\phi)\phi\log\phi - \Psi(\phi) \tag{22}$$

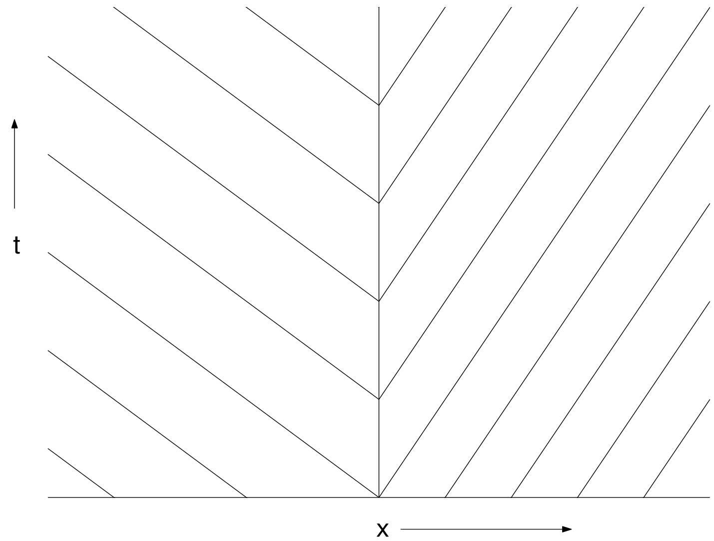
At interface.

- Integral entropy inequality gives entropy jump condition at interface

$$k_R F(\phi_+) - k_L F(\phi_-) \leq 0 \tag{23}$$

- This rules out the situation where

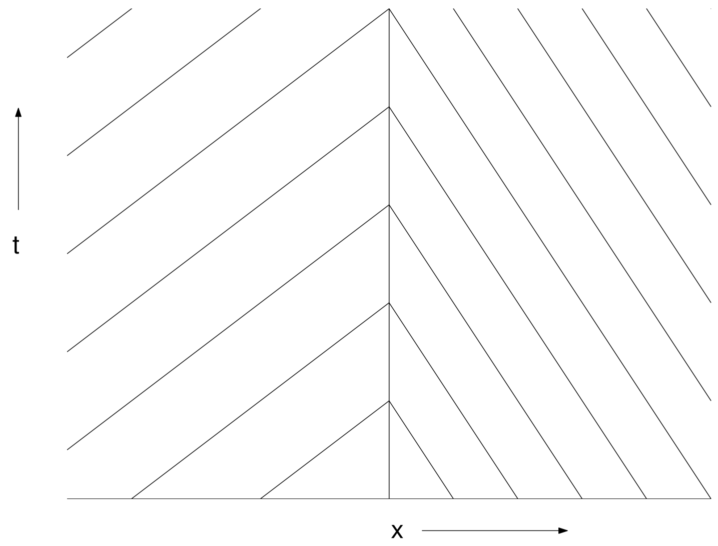
$$f'(\phi_-) < 0, \quad f'(\phi_+) > 0 \tag{24}$$



At interface.

- Our interface condition basically agrees with that of Zhang et. al.
- We allow one case not allowed by Zhang et. al.

$$f'(\phi_-) > 0, \quad f'(\phi_+) < 0 \tag{25}$$



- The question of whether the interface jump conditions are reasonable for traffic modeling has not been addressed

At interface.

- **Lemma:** Scheme satisfies discrete entropy inequality:

$$\Delta x \left(\frac{d}{dt} \mathcal{E}(\vec{\phi}_j) + D_+ \tilde{\mathcal{F}}_{j-\frac{1}{2}} \right) \leq 0, \tag{26}$$

where numerical entropy flux is modified slightly for variable k

$$\tilde{\mathcal{F}}_{j-\frac{1}{2}} := \mathcal{E}_{\vec{\phi}}(\vec{\phi}_j) \cdot \left(\vec{h}_{j-\frac{1}{2}} - k_j \vec{f}(\vec{\phi}_j) \right) + k_j \mathcal{F}(\vec{\phi}_j), \tag{27}$$

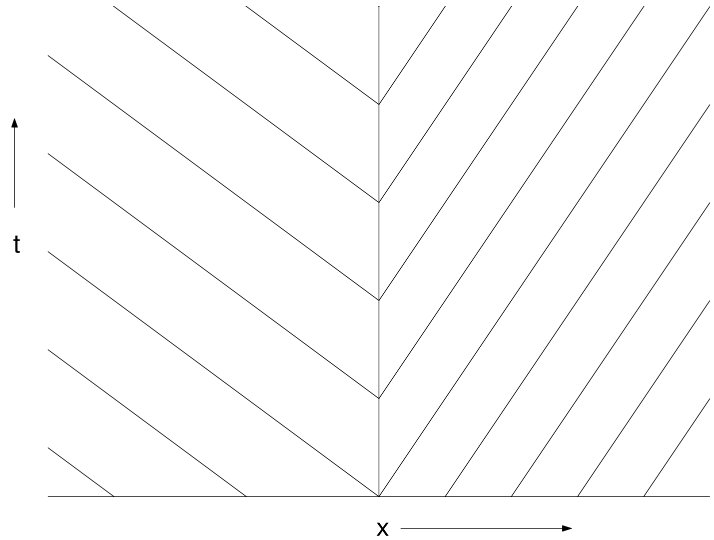
and

$$\Psi(\phi) = \begin{cases} \int_1^{\phi} \psi(z)\, dz, & k_L < k_R \\ \int_0^{\phi} \psi(z)\, dz, & k_L > k_R \end{cases} \tag{28}$$

At interface.

Theorem: For piecewise continuous limit solutions, the integral entropy inequality (with the two specific choices of Ψ) are satisfied. The **interface** entropy jump condition is satisfied, and we cannot have

$$f'(\phi_-) < 0, \quad f'(\phi_+) > 0 \tag{29}$$



Interface fix.

Interface fix.

• Scheme sometimes creates small spurious traveling waves originating at interface

• Modify numerical flux at one point (motivated by Rankine-Hugoniot condition):

$$h_{i,1/2}^{\mathrm{intfc}} := \min\left(h_{i,1/2}, \frac{\phi_{i,0}}{\phi_0}\min(k_L, k_R)V_i f_{\max}\right). \qquad (30)$$

• This eliminates the problem almost entirely

• Modified scheme keeps desirable properties:

1. Invariant region

2. Discrete entropy inequality

Interface fix.