Chapter 2.3
Problem #3 Variation
A tank originally contains 100 gal of fresh water. Then water containing \( \frac{1}{2} \) lb of salt per 2 gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. What is the amount of salt at any instant?

\[
\frac{dQ}{dt} = \text{rate in} - \text{rate out} \\
= (\text{salt concentration in}) \times (\text{flow rate in}) - (\text{tank salt concentration}) \times (\text{flow rate out})
\]

*Note \( \frac{dQ}{dt} \) should be in terms of \( \frac{\text{lb}}{\text{min}} \)

**Parameters**

- \( Q(t) \) : the amount of salt at time \( t \) (lbs)
- \( Q_o \) : initial amount of salt in the tank
- \( Q(0) = Q_o = 0 \)
- \( \text{salt concentration in} = \frac{1}{2} \text{ lb} \text{ gal}^{-1} \)
- \( \text{tank salt concentration} = \frac{Q}{100 \text{ gal}} \)

In this question flow rate in is the same as flow rate out so we will let the rates be defined as:

\[
\text{rate in} = \text{rate out} = 2 \frac{\text{gal}}{\text{min}}
\]

**Substituting everything into our differential equation, we arrive at**

\[
\frac{dQ}{dt} = \frac{1}{2} \frac{\text{lb}}{\text{gal}} \cdot 2 \frac{\text{gal}}{\text{min}} - \frac{Q}{100 \text{ gal}} \cdot 2 \frac{\text{gal}}{\text{min}}
\]

\[
= \frac{1}{2} \frac{\text{lb}}{\text{min}} - \frac{Q}{50 \text{ min}}
\]

**Rewriting our equation, we have**

\[
\frac{dQ}{dt} + \frac{Q}{50} = 1
\]
Multiplying by the integrating factor $e^{t/50}$ followed by applying the product rule for derivatives we have

$$\frac{dQ}{dt} \cdot e^{t/50} + Q \cdot \frac{e^{t/50}}{50} = 1 \cdot e^{t/50}$$

$$\frac{d}{dt}[Q \cdot e^{t/50}] = e^{t/50}$$

$$\int \frac{d}{dt}[Q \cdot e^{t/50}]dt = \int e^{t/50}dt$$

After integrating both sides we are left with

$$Q \cdot e^{t/50} = 50 \cdot e^{t/50} + C$$

$$Q = 50 + Ce^{-t/50}$$

From our initial condition $Q(0) = 0$ we have $C = -50$. Our solution to this differential equation is

$$Q(t) = 50 - 50e^{-t/50}$$

What is the amount of salt after 10 minutes?

$$Q(10) = 50 - 50e^{-10/50}$$

$$\approx (9.063)$$

What is the amount of salt after 30 minutes?

$$Q(10) = 50 - 50e^{-30/50}$$

$$\approx (22.559)$$
Chapter 2.3
Problem #4
A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow...

The first thing to note is now the rate out does not match the rate in. The amount of the solution in the tank is increasing (3 gal/min IN - 2 gal/min OUT = INCREASING by 1 gal/min). This alters our equation for the tank salt concentration. It will now be

\[
\text{tank salt concentration} = \frac{Q}{\text{current amount of water solution}} = \frac{Q}{200 + t}
\]

\[
\frac{dQ}{dt} = 1 \text{ lb/gal} \cdot 3 \text{ gal/min} - \frac{Q}{200 + t} \text{ lb/gal} \cdot 2 \text{ gal/min}
\]

\[
= 3 \text{ lb/min} - \frac{2Q}{200 + t} \text{ lb/min}
\]

Rewriting our equation, we have

\[
\frac{dQ}{dt} + \frac{2}{200 + t} Q = 3
\]

with the initial condition of \( Q(0) = 100 \).

Multiplying by the integrating factor \((200 + t)^2\) followed by applying the product rule for derivatives we have

\[
\frac{dQ}{dt} \cdot (200 + t)^2 + \frac{2}{200 + t} Q \cdot (200 + t)^2 = 3 \cdot (200 + t)^2
\]

\[
\frac{dQ}{dt} \cdot (200 + t)^2 + Q \cdot 2(200 + t) = 3(200 + t)^2
\]

\[
\frac{d}{dt}[Q \cdot (200 + t)^2] = 3(200 + t)^2
\]

After integrating both sides we are left with

\[
Q \cdot (200 + t)^2 = (200 + t)^3 + C
\]

\[
Q = (200 + t) + C(200 + t)^{-2}
\]

From the initial condition, \( Q(0) = 100 \), we get \( C = -100(200)^2 \). Our solution to this differential equation is

\[
Q(t) = 200 + t - 100(200)^2(200 + t)^{-2}
\]

\[
= 200 + t - \frac{100(200)^2}{(200 + t)^2}, \quad t < 300
\]

We have to consider when the tank will begin to overflow which is after 300 min since we initially have 200 gal in our 500 gal tank which leads to, \( t < 300 \).
As for, “Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.”

We let \( t = 300 \) which gets us \( Q(300) = 484 \text{lb of salt} \), but since it asks for the concentration we take our solution and divide it by the amount of water in the tank and arrive at

\[
\text{Concentration} = \frac{Q(t)}{200 + t} = \frac{200 + t - \frac{100(200)^2}{(200+t)^3}}{200 + t}
\]

\[
= 1 - \frac{100(200)^2}{(200 + t)^3}
\]

Evaluate when \( t = 300 \)

\[
= \frac{484\text{lb}}{500\text{gal}} = 121/125 \text{ lb gal}
\]

If the tank had infinite capacity the concentration would then converge to,

\[
\lim_{t \to \infty} \frac{Q(t)}{200 + t} = \lim_{t \to \infty} \left( 1 - \frac{100(200)^2}{(200 + t)^3} \right)
\]

\[
= \frac{1 \text{ lb}}{\text{gal}}
\]
Tank with salt coming in from two pipes:
A tank with a capacity of 1500 gals originally contains 1000 gals of fresh water. The first pipe containing \( \frac{1}{2} \) lb of salt per gallon is entering at a rate of 4 gal/min. The second pipe containing \( \frac{1}{3} \) lb of salt per gallon is entering at a rate of 6 gal/min. The mixture is allowed to flow out of the tank at a rate of 5 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

Note* the rate out does not match the rate in. The amount of the solution in the tank in increasing (10 gal/min IN - 5 gal/min OUT = INCREASING by 5 gal/min). This alters our equation for the tank salt concentration. It will now be

\[
\text{tank salt concentration} = \frac{Q}{\text{current amount of water solution}} = \frac{Q}{1000 + 5t}
\]

Determine how much salt is coming in

Pipe 1 = \( \frac{1}{2} \) lb \( \frac{4 \text{ gal}}{\text{min}} \)
Pipe 2 = \( \frac{1}{3} \) lb \( \frac{6 \text{ gal}}{\text{min}} \)
Total Rate In = Pipe 1 + Pipe 2
\[
= \frac{2 \text{ lb}}{\text{min}} + \frac{2 \text{ lb}}{\text{min}} = 4 \frac{\text{lb}}{\text{min}}
\]

Determine how much salt is going out

Rate Out = \( \frac{Q}{1000 + 5t} \) \( \frac{\text{lb}}{\text{gal}} \) \( \frac{5 \text{ gal}}{\text{min}} \)
\[
= \frac{5Q}{1000 + 5t} \frac{\text{lb}}{\text{min}}
\]

Combining everything our differential equation becomes

\[
\frac{dQ}{dt} = 4 \frac{\text{lb}}{\text{min}} - \frac{5Q}{1000 + 5t} \frac{\text{lb}}{\text{min}}
\]

\[
\frac{dQ}{dt} + \frac{5Q}{1000 + 5t} = 4
\]

Multiplying by our integrating factor \( 1000 + 5t \) we arrive to

\[
\frac{dQ}{dt} \cdot (1000 + 5t) + \frac{5Q}{1000 + 5t} \cdot (1000 + 5t) = 4 \cdot (1000 + 5t)
\]
\[
\frac{d}{dt}[Q \cdot (1000 + 5t)] = 4 \cdot (1000 + 5t)
\]
\[
Q \cdot (1000 + 5t) = 4(1000t + \frac{5t^2}{2}) + C
\]
\[
Q = \frac{4(1000t + \frac{5t^2}{2}) + C}{1000 + 5t}
\]
Because \( Q(0) = 0 \), we get \( C = 0 \) which makes our solution

\[
Q = \frac{4(1000t + \frac{5t^2}{2})}{(1000 + 5t)}, \quad t < 100
\]

Since our tank overflows after 1500 gallons at time \( t = 100 \), to find the amount of salt at that instant we evaluate \( Q(100) \).

\[
Q(100) = \frac{500000}{1500} = \frac{1000}{3} \approx 333.333
\]