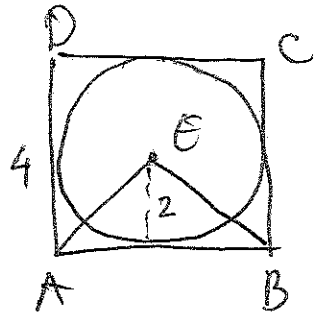
AMATYC – Spring 2010

1) $P(4)=32+8=40$, $P(2)=2$
$P(4)-P(2)=38=2\cdot 19$ B: 19

2) 

$\frac{1}{4}(A_{SQ}-A_{CIR})=\frac{1}{4}(16-4\pi)=4-\pi$ C: $4-\pi$

3) $2a+b=10$
$3b+a=8 \to a=8-3b$ } $16-5b=10$
$b=1.2$, $a=\ldots$
$a+b=5.6$ B: 5.6

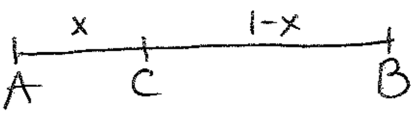
4) $\frac{x+1}{x-3}-2\geq 0 \to \frac{x+1-2x+6}{x-3}\geq 0$ E: $(3,7]$
$\to \frac{7-x}{x-3}\geq 0$

$-\ (\ +++\ ]\ -$
$\quad 3 \qquad 7$

5) $2+3d=(2+d)^2-8$ C: 8
$d^2+d-6=0$ [since $d>-1$] $\Rightarrow d=2$
$a_3=2+3d=8$

6) Test using calculator: $\sqrt{2010-10^3-1^3}$, $\sqrt{2010-9^3-2^3}$, …, $\sqrt{2010-5^3-6^3}$
None works. So A) is not true.
The same for B, C when we try D: $a+b=14$ we get D: 14
$\sqrt{2010-5^3-9^3}=34$. So $5^3+9^3+34^2=2010$, $a+b=14$

7) $z^2=21-20i=25-20i+4i^2=(5-2i)^2$ A: 7
$z=5-2i$, $|a|+|b|=7$

8) 

$\frac{x}{1-x}=\frac{1-x}{5} \to 5x=1-2x+x^2$
$x^2-7x+1=0$
$x=\frac{7\pm\sqrt{45}}{2}$, $x<1 \to x=\frac{7-\sqrt{45}}{2}$

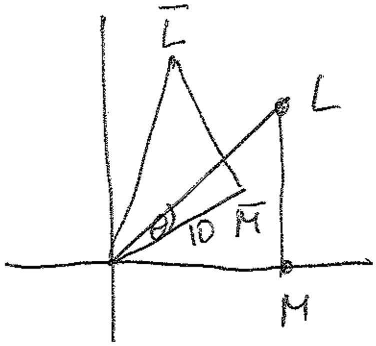
$\frac{AC}{BC}=\frac{x}{1-x}=\frac{1-x}{5}=\frac{1}{5}\left[1-\frac{7-3\sqrt{5}}{2}\right]=\frac{3\sqrt{5}-5}{10}$ A: $\frac{3\sqrt{5}-5}{10}$

9) $n=1,\ldots,4$ $\lfloor \log_5 n\rfloor=0$
$n=5,\ldots,24$ $\lfloor \log_5 n\rfloor=1$
$n=25,\ldots,124$ $\lfloor \log_5 n\rfloor=2$
$n=125,\ldots,624$ $\lfloor \log_5 n\rfloor=3$
$n=625,\ldots,2010$ $\lfloor \log_5 n\rfloor=4$
$S=20\cdot 1+100\cdot 2+500\cdot 3+1386\cdot 4=7264$
E: 7264

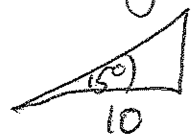
(10) 112, 113, 115, 121, 131, 151, 211, 311, 511

$P(A) = \frac{9}{6\cdot 6\cdot 6} = \frac{1}{24}$ B: $\frac{1}{24}$

(11)



Overlapping triangle



$A = \frac{10\cdot 10 \tan 15^\circ}{2} = 50 \tan 15^\circ$

$A = \frac{10\cdot 10}{2}\cdot 2 - 50\tan 15^\circ \approx 86.6$ E: 87

(12) $ab = 48$
$ac = 50$
$bc = 54$

$V = abc = \sqrt{48\cdot 50\cdot 54} = 360$ A: 360

(13)

$2010 = 2\cdot 3\cdot 5\cdot 67$

Every factor of 2010 has form: $2^{k_1}3^{k_2}5^{k_3}67^{k_4}$ where k_i's are either 0's or 1's. There are 16 of them. 8 of them have $k_4 = 1$, 8 of them $k_4 = 0$ and so on. When we multiply them we get $2^8\cdot 3^8\cdot 5^8\cdot 67^8$. This is product of all 4 columns. Since these products are equal, each of them is $(2^8 3^8 5^8 67^8)^{\frac{1}{4}}$

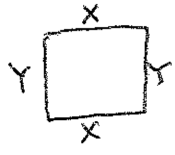
$= 2010^2 = \boxed{4040100.}$

(14) $f(f(x)) = \sqrt{\dfrac{\frac{x^2+1}{x^2-1}+1}{\frac{x^2+1}{x^2-1}+1}} = \sqrt{\frac{2x^2}{2}} = |x| = g(x)$

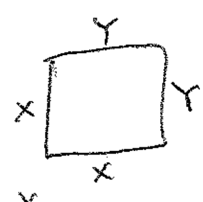
$f^{2010}(x) = (f^2(x))^{1005} = g^{1005}(x) = |x|$ B: $|x|$

(15) 1 color: $\underline{3}$ RRR, WWW, BBB

2 colors:

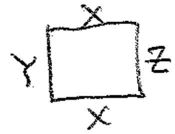


X – 3 choices
Y – 2 choices
$3\cdot 2 = \underline{6}$

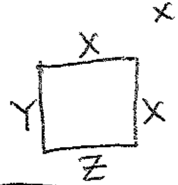


$3\cdot 2 = \underline{6}$ Total: $\underline{12}$

3 colors:



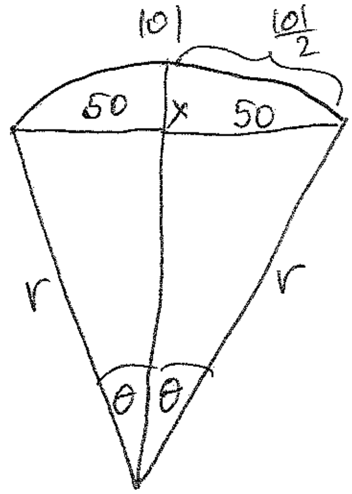
$\underline{\text{3 choices}}$



$3\cdot 2 = \underline{6}$ Total: $\underline{9}$
Order of Y, Z matters.

$3 + 12 + 9 = 24$ B: 24

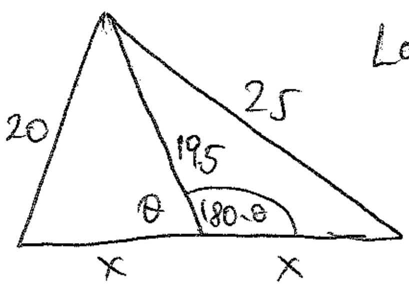
(16)



$r\theta = \frac{101}{2}$
$r\sin\theta = 50$ $\Big\}$ $\frac{\sin\theta}{\theta} = \frac{100}{101}$ $\xrightarrow{\text{calculator}}$ $\theta \approx 0.244$, $r \approx 207$

$x = 207 - 207\cos\theta \approx 6.13$ **D: 6**

(17)



Law of cosines: $20^2 = x^2 + 19.5^2 - 39x\cos\theta$

$25^2 = x^2 + 19.5^2 + 39x\cos\theta$

Add: $20^2 + 25^2 = 2x^2 + 2\cdot 19.5^2$

$x = \frac{1}{2}\sqrt{400 + 625 - 2\cdot 19.5^2} = 11.5$

$2x = 23$ **B: 23**

(18) $abba = 1001a + 110b \equiv_7 5b$ because $1001 = 7\cdot 143$, $110 = 7\cdot 15 + 5$
To be divisible by 7, b has to be 0 or 7. $p = \frac{2}{10} = \frac{1}{5}$ **D: $\frac{1}{5}$**

(19) $m^2 = (a+b)^2 = a^2 + 2ab + b^2 = n + 2ab \rightarrow \frac{m^2 - n}{2} = ab$

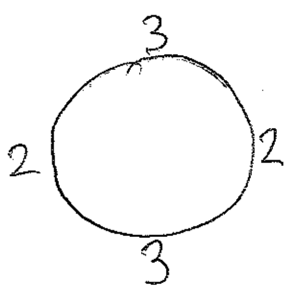
$a^3 + b^3 = (a+b)^3 - 3ab(a+b) = m + n$

$m^3 - 3m\frac{m^2 - n}{2} = m + n \rightarrow 2m^3 - 3m^3 + 3mn = 2m + 2n \rightarrow n = \frac{m^3 + 2m}{3m - 2}$

$3m-2 \mid m^3 + 2m \rightarrow 3m-2 \mid 9m^3 + 18m$
$3m-2 \mid 9m^2 - 4 \rightarrow 3m-2 \mid 9m^3 - 4m$ $\Big\}$ subtract $3m-2 \mid 22m \rightarrow 3m-2 \mid 66m$
$3m-2 \mid 66m-44$

$3m-2 \mid 44 \rightarrow 3m-2 \in \{44, 22, 11, 4, 2, 1\} \rightarrow 3m \in \{46, 24, 13, 6, 4, 3\}$ the highest value for $3m$ is 24 so $m = 8 \rightarrow n = 24$. **D: 24**

(20)



He needs to get a symmetric position accross the circle (and keep it that way).
To get there he needs the move: 2w|s

A: 2w|s