# AMATYC – Spring '11

(1) $0.8E = 2 \cdot 0.6A \rightarrow E = 1.5A \rightarrow$ Answer: D: 150

(2) $a\#b = a(b^2+1)$

$(a\#b)\#3 = a(b^2+1)(3^2+1) = 250 \rightarrow a(b^2+1) = 25$

$\rightarrow b^2+1 = 5$ (integer $>1$, $<25$, square+1)

$\left.\begin{array}{l}\rightarrow b = 2 \\ \rightarrow a = 5\end{array}\right\} a+b = 7$ B: 7

(3) Let $C_n$ denote # of possible ways to climb $n$ steps. She can get there finishing with step of length 1, 2 or 4. So:

$C_n = C_{n-1} + C_{n-2} + C_{n-4}$.

It is easy to see: $C_1 = 1$, $C_2 = 2$, $C_3 = 3$, $C_4 = 6$. Using recursion we get $C_5 = 10$, $C_6 = 18$, $C_7 = 31$, $C_8 = 55$, $C_9 = 96$, $C_{10} = 169$. E: 169

(4) $n + (n+1) + \cdots + (n+5) = 6n + 15 = x^3 \rightarrow 3(2n+5) = x^3 \rightarrow x = 3y$

$3(2n+5) = 27y^3 \rightarrow 2n+5 = 9y^3$. $y$ must be odd. $y = 1$ gives $n = 2$ do not count.

$\left.\begin{array}{l} y = 3 \rightarrow 2n+5 = 81 \rightarrow n = 119 \\ y = 5 \rightarrow 2n+5 = 1125 \rightarrow n = 560 \end{array}\right\}$ Sum: 679 A: 679

(5) $\frac{a}{1-q} = 6$, $\frac{a^2}{1-q^2} = 15$ Divide: $\left(\frac{a}{1-q}\right)^{6} \cdot \frac{a}{1+q} = 15$

$\frac{a}{1+q} = \frac{15}{6} = \frac{5}{2}$ B: 2.5

(6) Let the 5-th number be $a$ and 6-th number $b$, $a \le b$.

$\frac{a+b}{2} = 6$. When 15 is added $b$ becomes the middle: $b = 8$

So $a = 4$. If 7 is added then 7 is in the middle. E: 7

(8) $SL > \sqrt{SM^2 + ML^2} = \sqrt{433} \approx 20.81$. SL integer: $SL \ge 21$ C: 21

(9) $p(x) = 3 + ax + bx^2 + cx^3 + dx^4$

$p(1) = 8 \rightarrow$
$a + b + c + d = 5$
$a + 2b + 4c + 8d = 18$
$a + 3b + 9c + 27d = 47$
$\cdot(-1)$ $\cdot(-1)$

$\left.\begin{array}{l} b + 3c + 7d = 13 \\ b + 5c + 19d = 29 \end{array}\right\} c + 6d = 8$

$a, b, c, d$ integers: $d = 0$ or $d = 1$. $d = 0 \rightarrow c = 8$, but $c \le 5$ #. So $d = 1$.

Now, $c = 2$, $b = 0$, $a = 2$.

$p(-2) = 3 - 4 - 16 + 16 = -1$ E: $-1$

(10) No 2's allowed after place 2 (appropriate 3-cut makes it composite).
The lowest we can go (to minimize) is pattern: 113113! ~~because~~ 131, 311
are both prime. Answer: 113113113̲1 Answer: E: 31

(11) $a, a+d, a+2d, \ldots$ $b, bq, bq^2, \ldots$ Let $d = \frac{1}{b}$, $r = \frac{1}{q}$

$\frac{1}{b} = z$, $\frac{1}{bq} = zr$, $\frac{1}{bq^2} = zr^2$, $\frac{1}{bq^3} = zr^3$. So we have

$ab = 96$
$(a+d)bq = 180$
$(a+2d)bq^2 = 324$
$(a+3d)bq^3 = 567$

$a = 96z$
add $a+d = 180zr$
$a+2d = 324zr^2$ Add
$a+3d = 567zr^3$

$96z + 567zr^3 = 180zr + 324zr^2$
$567r^3 - 324r^2 - 180r + 96 = 0$
Calculator: $r = \frac{2}{3} \to q = \frac{3}{2}$

$a = 96z$
$a+d = 120z \to d = 24z \to a = 4d$

so $(a+4d)bq^4 = (96z+96z)b\left(\frac{3}{2}\right)^4 = 2a \cdot b \frac{81}{16} = \frac{81}{8} ab = \frac{81}{8} \cdot 96 = 972$: E

(12) Let $\log_x y = a$. We have $a + \frac{1}{a} = 2.9$. Quadratic, $a > 0$
$a = 2.5$. So $\log_x y = 2.5 \to y = x^{\frac{5}{2}}$, $xy = x^{\frac{7}{2}} = 128 \to x = 4$.
$y = 4^{\frac{5}{2}} = 32 \to x+y = 36$ B

(13) $a^5 + b^2 + c^2 = 2011$. $[\sqrt[5]{2011}] = 4$, so $a = 1, 2, 3$ or $4$. Since non-
primes are given as potential solutions, $a \neq 1$, $a \neq 4$. So $a = 2$ or $a = 3$
Test the expression $\sqrt{2011 - 2^5 - c^2}$ (for C: one of the answers offered)
in calculator. None works. Now check $\sqrt{2011 - 3^5 - c^2}$. Works for $c = 42$
we get answer 2. So $2011 = 3^5 + 42^2 + 2^2$. Answer non-prime C: 42

(14) $\overline{abba} = a \cdot 1001 + 10 \cdot \overline{bb} = a \cdot 1001 + 110b \equiv_{17} 8b - 2a$. To be divisible
by 17 we need $4b - a$ to be divisible by 17.
$4b - a = 0 \to b = 4a$ gives $(1,4), (2,8)$ Total count: C: 5
$4b - a = 17 \to a = 3, a = 7$ so $(3,5), (7,6)$
$4b - a = 34 \to a = 2, b = 9$ so $(2,9)$

(15) There are 7 numbers, one is dropped. Doesn't matter which. We
will count it for 0, 1, 2, 3, 4, 5 and multiply answer by 7

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 3 | 4 | 5 |

| | | |
|---|---|---|
| 0 | 1 | 3 |
| 2 | 4 | 5 |

| | | |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 4 | 5 |

| | | |
|---|---|---|
| 0 | 1 | 4 |
| 2 | 3 | 5 |

| | | |
|---|---|---|
| 0 | 2 | 4 |
| 1 | 3 | 5 |

$7 \cdot 5 = 35$: D

(16) $a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b = 160$. $b > a \to 13b > 160$
Since $5 | b$ and $b < 20$, we have $b = 15$ and $a = 8$
$a_8 = 8a + 13b = 64 + 195 = 259$ C: 259

(17) $d \mid n^2+7, n+4 \to d \mid n^2+7, (n+4)(n-4) \to d \mid n^2+7-(n^2-16) \to d \mid 23$

So $d=1$ or $d=23$. We only need factors $d>1$ so $d=23$.

$23 \mid n+4 \to n = 23k-4$ for $k=1,\dots,\left[\frac{2011+4}{23}\right]=87$. Answer C: 87

(18) Let $P_n$ be the answer for problem with $n$ piles (here $n=10$).

$$P_{n+1} = P_n \cdot \frac{2n}{2n+1} + (1-P_n)\cdot\frac{1}{2n+1}$$

(If it was odd to keep it we need dime. If it was even to make it odd we need a penny)

$$= P_n + \frac{1-2P_n}{2n+1}$$

$n=1 \to \frac{1}{3}$, $n=2 \to \frac{1}{3}\cdot\frac{3}{5}+\frac{1}{5}=\frac{2}{5}$

$n=3$ $\frac{6}{15}\cdot\frac{5}{7}+\frac{1}{7}=\frac{3}{7}$ Recognize the pattern $P_n = \frac{n}{2n+1}$ A: $\frac{10}{21}$

$n=4$ $\frac{3}{7}\cdot\frac{7}{9}+\frac{1}{9}=\frac{4}{9}$ $P_{10}=\frac{10}{21}$

(19) The lowest value for the sum is 15. Try with 15: D: 8

$\{15\}$ $\{1,14\}$, $\{2,13\}$, ..., $\{7,8\}$. Total: 8 sets

(20) 4 primes, 5 composites. Wherever I put prime I will put 1, composite 0.
Every configuration of 0's and 1's could be filled with "alive" numbers
in $5!\cdot 4!$ ways. Let us count configurations. I will call 10 block +, 01 —
Count those that start with 0. There are 4 0's and 4 1's remaining.
According to pigeon-hole principle: 0 ☐☐☐☐ every box must
have exactly one 1 and one 0. Also, to avoid two 1's next to each
other after any 01, all the others must be 01 i.e. —.
So we can have 0++++, 0+++−, 0++−−, 0+−−−, 0−−−−
This is 5 configurations starting with 0. There are also 5 ending with 0.
Overlap? 0 ☐ 0. Yes only one: 0 10 10 10 10. So total
number of configurations starting or ending with 0 is $5+5-1=9$.
Now those that start and end in 1. This is: 10 ☐☐☐☐☐ 01 There are
$\binom{5}{2}$ of those. Remove those that have 11: 11000, 01100, 00110, 00011 $10-4=6$
Total: 15 configurations.

$$P(A) = \frac{15\cdot 5!\cdot 4!}{9!} = \frac{15\cdot \cancel{24}^{3}\cdot \cancel{5!}}{9\cdot 8\cdot 7\cdot 6\cdot \cancel{5!}} = \frac{45}{9\cdot 42} = \frac{5}{42}$$ C: $\frac{5}{42}$

(7) If you color the board like chess board, both removed corners will have the same color (e.g. black). If we color 1x2 domino as black-white the covering would have 24B, 24W and board has 23B, 25W. So 1x2 (~~hence~~ 2x2, 3x2) can not be done.